6.1 – Resonators Eigenvalues, Eigenmodes

ME-426 – Micro/Nanomechanical Devices

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Introduction

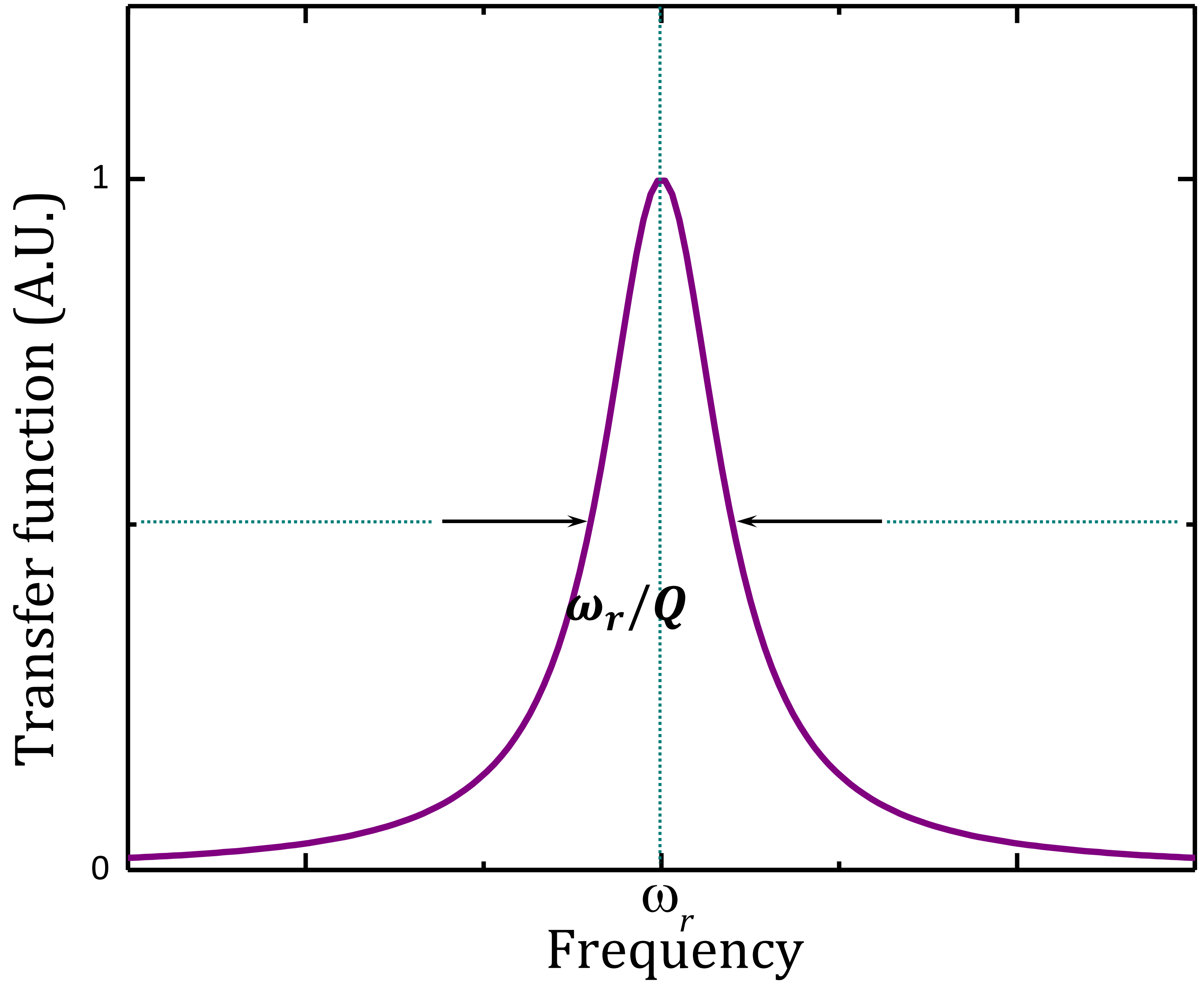
- Concept of resonator
- Eigenvalues – Resonant frequencies
- Eigenvectors – Normal modes

What is a resonator?

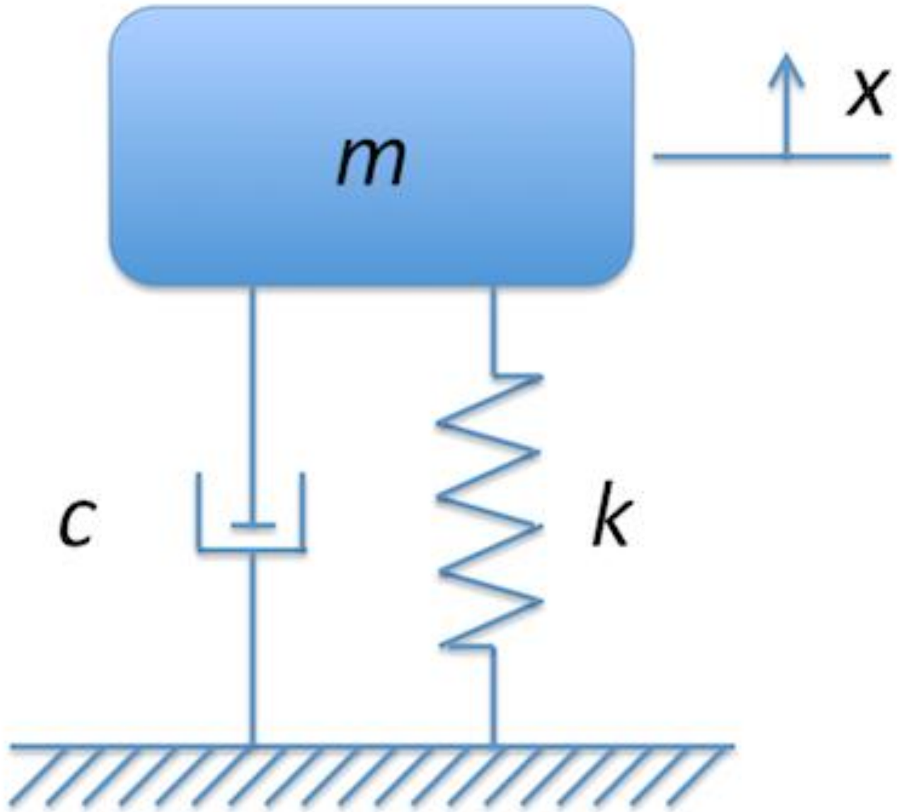
What is a resonator?

- System which response to an external force is amplified at given frequencies

- Enhancement determined by the quality factor

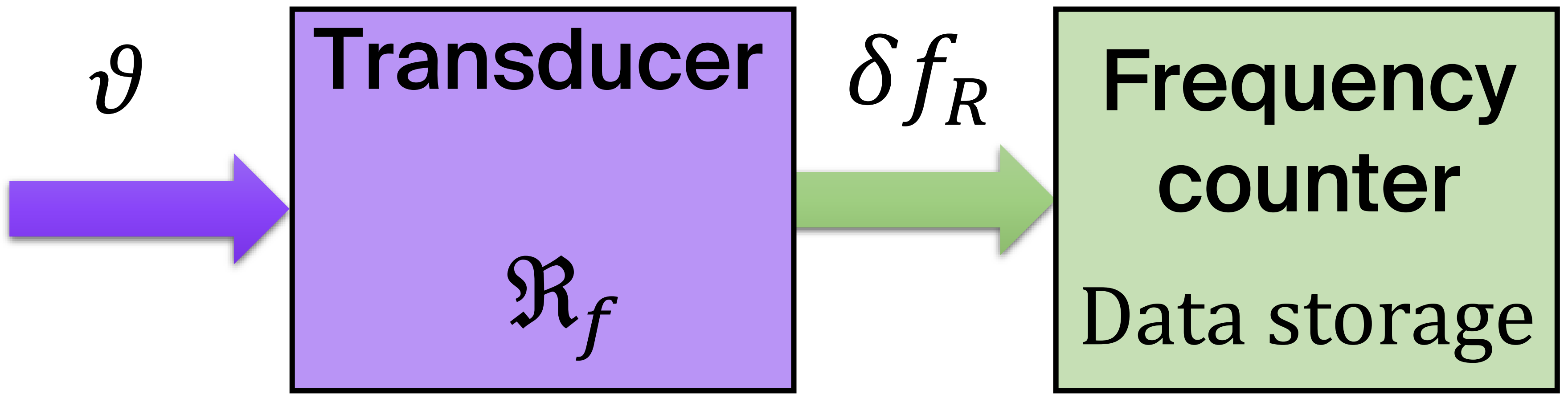


$$m\ddot{x} + c\dot{x} + kx = F(t)$$



Why resonators?

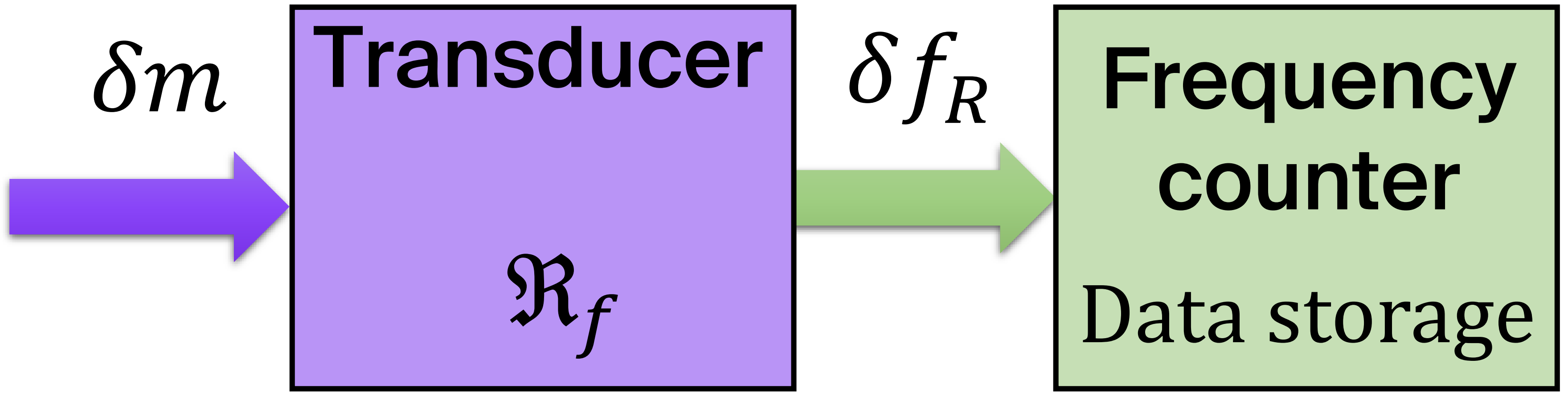
- Frequency is the magnitude that we can measure the most precise in the world



- Frequency stores information about different magnitudes
  - Mass
  - Stiffness
  - Temperature
  - Pressure

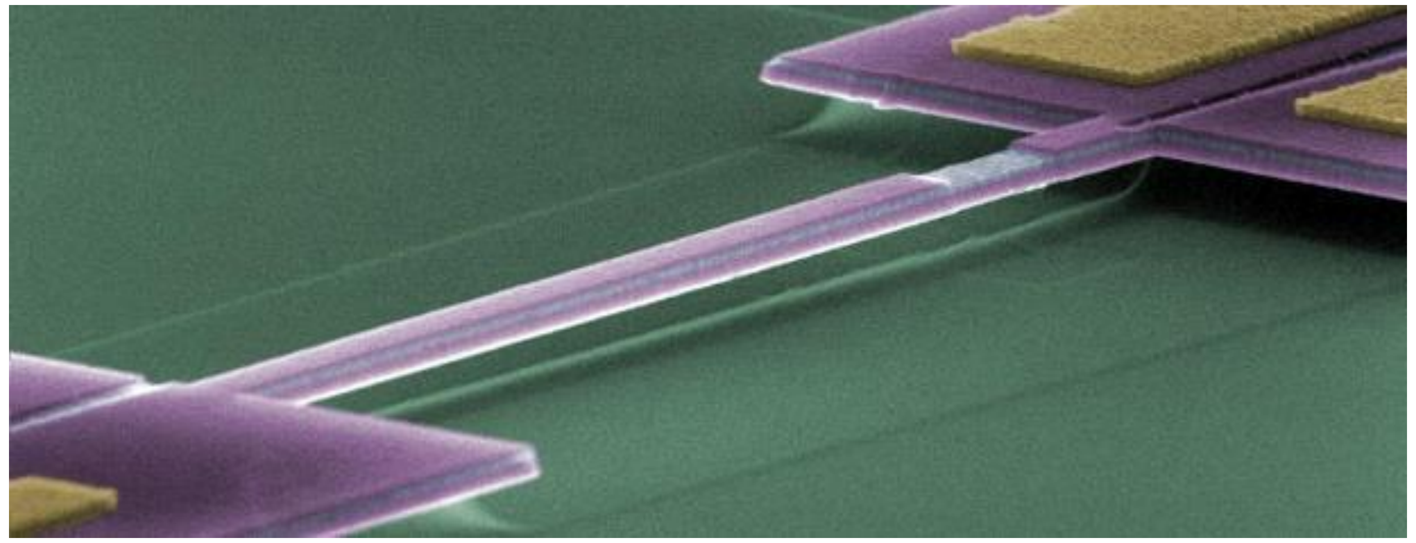
Why resonators? E.g. mass detection

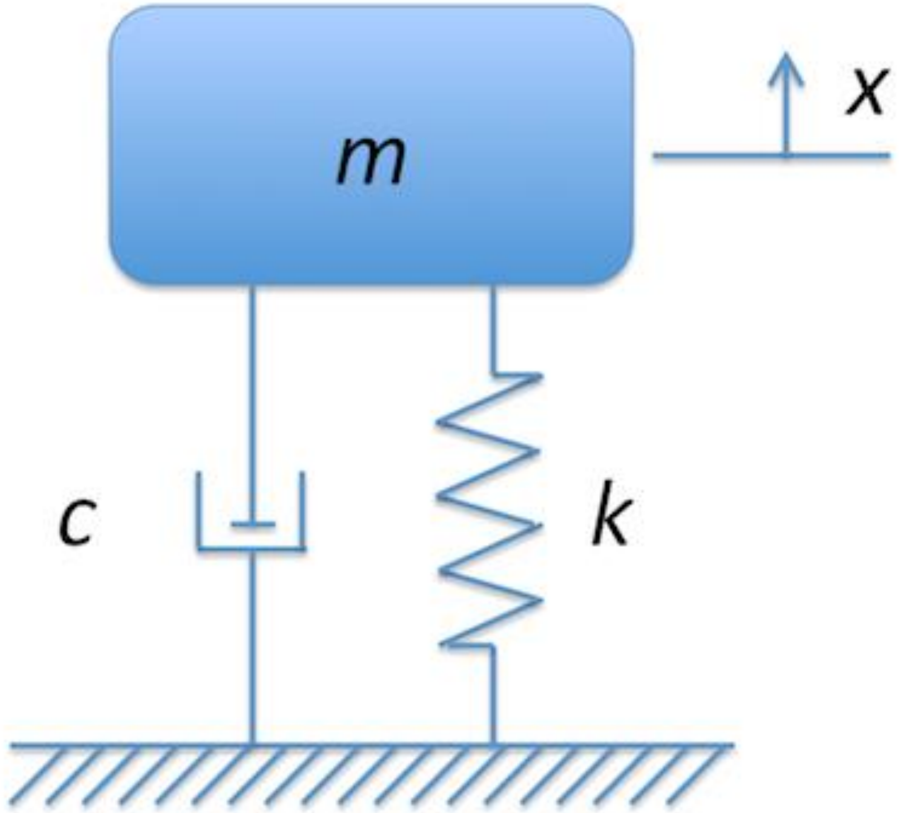
- Smaller mass means larger (relative) responsivity



$$f_R = \frac{1}{2\pi}\sqrt{\frac{k}{m}} \rightarrow \frac{\delta f_R}{f_R} = -\frac{1}{2}\frac{\delta m}{m}$$

$$m\ddot{x} + c\dot{x} + kx = F(t)$$

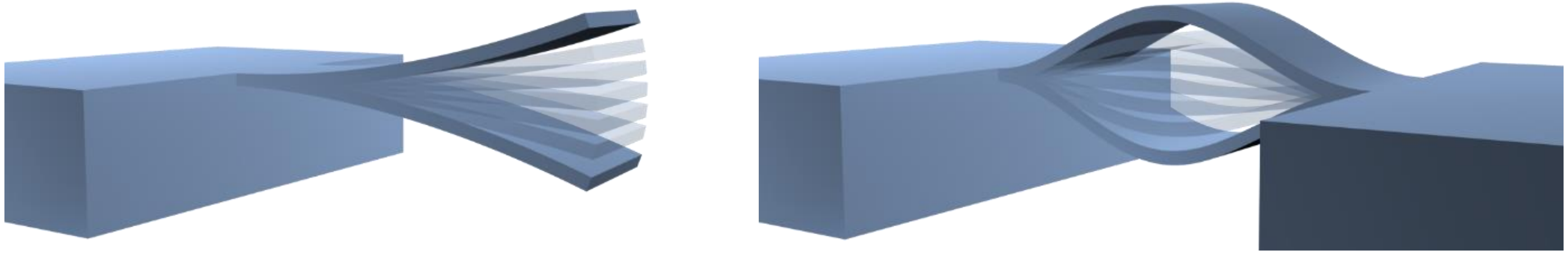




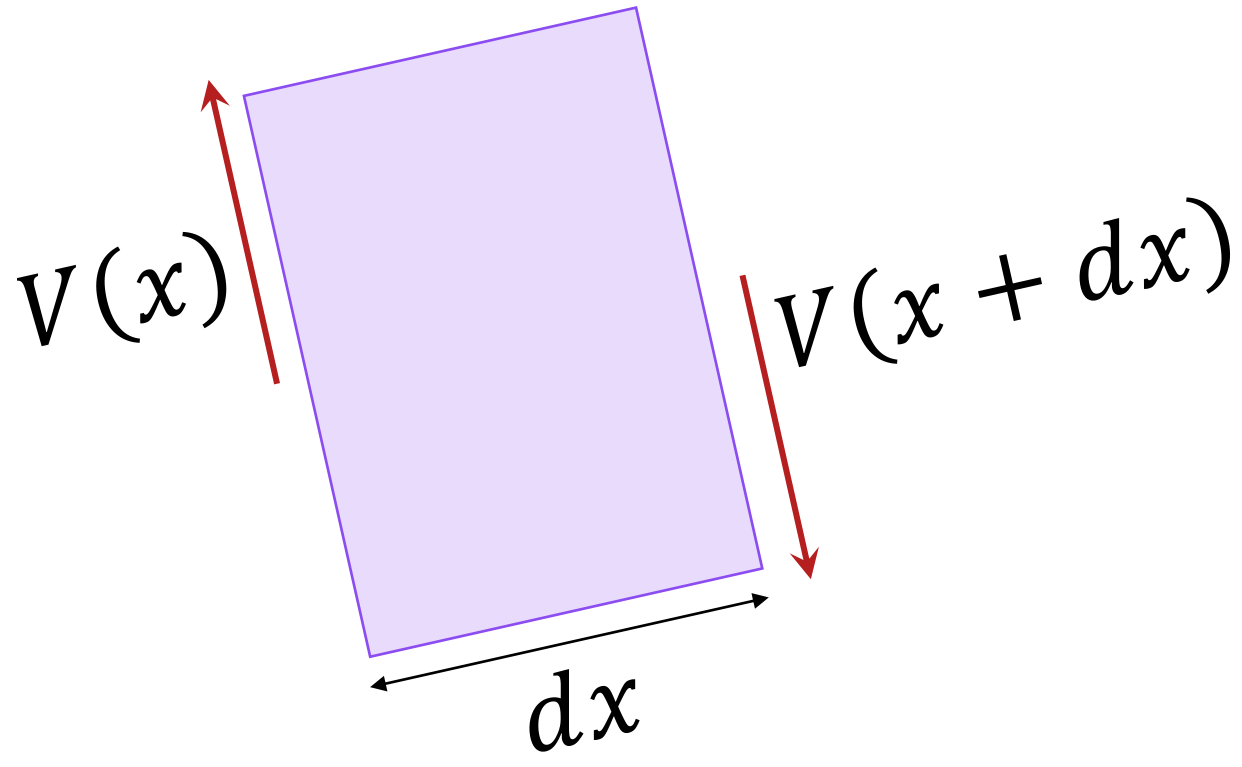
Eigenvalues

Solving the Eigenvalue problem

- We will consider flexural resonators



- If we take a differential section and we apply 2nd Newton's law:



Solving the Eigenvalue problem

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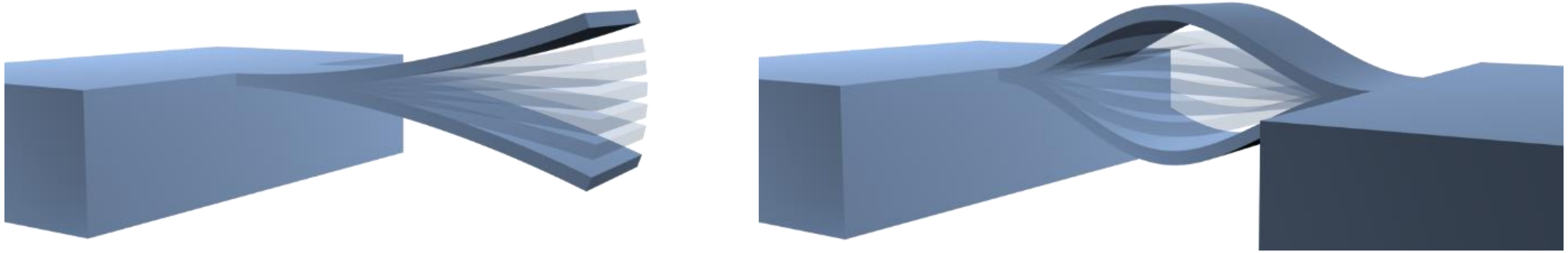
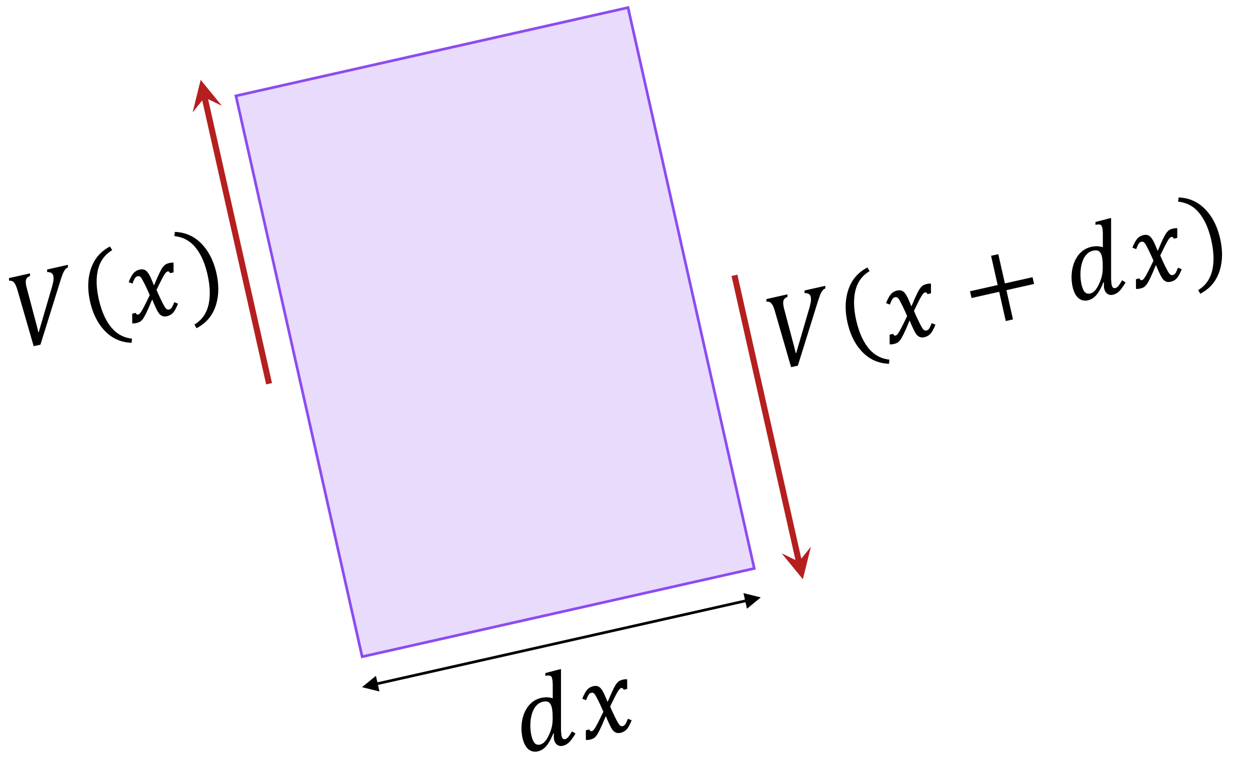


- If we take a differential section and we apply 2nd Newton's law:

$$F = ma \rightarrow V(x) - V(x+dx) = -\frac{\partial V}{\partial x}dx = \rho A\, dx \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial V}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} \rightarrow \boldsymbol{\frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 w}{\partial x^2}\right) = -\rho A \frac{\partial^2 w}{\partial t^2}}$$

$$\boldsymbol{\frac{\partial^4 w}{\partial x^4} = -\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}}$$



Solving the Eigenvalue problem

- We will consider flexural resonators
  - with uniform, monomaterial cross section (for the moment)

$$\frac{\partial^4 w}{\partial x^4} = -\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}$$

- Separation of variables, solution of the form: $w(x,t) = X(x)T(t)$

Solving the Eigenvalue problem

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- Separation of variables, solution of the form: $w(x,t) = X(x)T(t)$

$$\frac{1}{X(x)}\frac{\partial^4 X(x)}{\partial x^4} = -\frac{\rho A}{EI}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = constant = \frac{1}{\lambda^4}$$

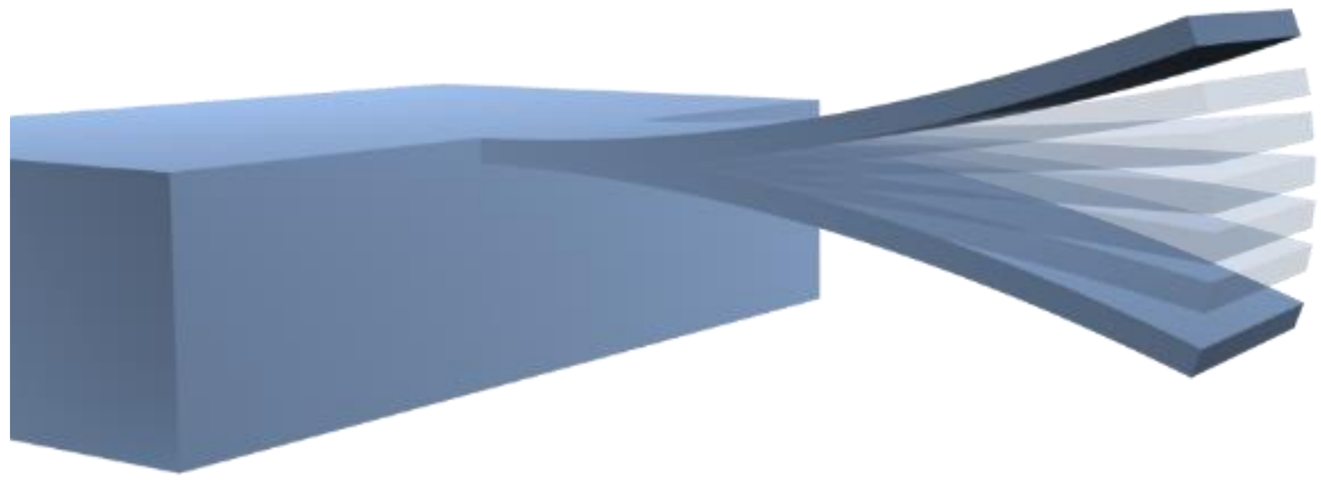
$$-\frac{\rho A}{EI}\frac{1}{T(t)}\frac{\partial^2 T(t)}{\partial t^2} = \frac{1}{\lambda^4} \rightarrow T(t) = B_1\cos(\omega t) + B_2\sin(\omega t)\,;\, \boldsymbol{\omega^2} = \boldsymbol{\frac{EI}{\rho A}\frac{1}{\lambda^4}}$$

$$\frac{1}{X(x)}\frac{\partial^4 X(x)}{\partial x^4} = \frac{1}{\lambda^4} \rightarrow X(x) = C_1\cos\left(\frac{x}{\lambda}\right) + C_2\cosh\left(\frac{x}{\lambda}\right) + C_3\sin\left(\frac{x}{\lambda}\right) + C_4\sinh\left(\frac{x}{\lambda}\right)$$

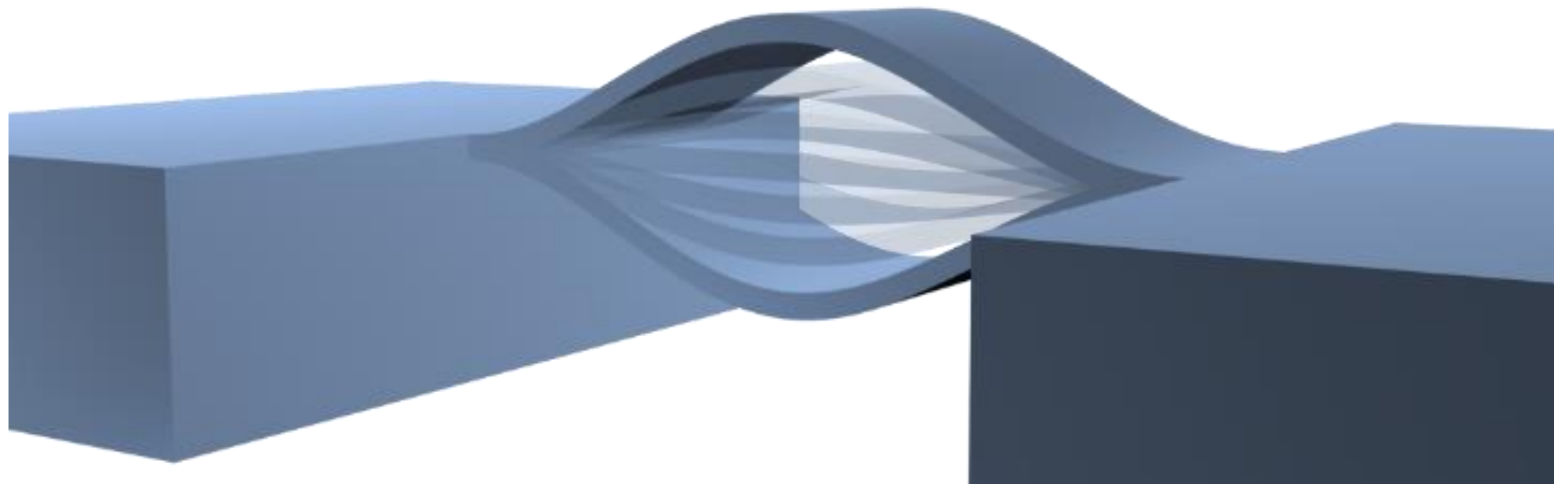
Eigenvalues – Boundary conditions

- Solution at $t = 0$ is maximum: $T(t) = B_1 \cos(\omega t)\,; \boldsymbol{\omega^2} = \frac{\boldsymbol{EI}}{\boldsymbol{\rho A}}\frac{\mathbf{1}}{\boldsymbol{\lambda^4}}$

- Spatial boundary conditions:



$$X(x = 0) = 0$$
$$X'(x = 0) = 0$$
$$X''(x = L) = 0$$
$$X'''(x = L) = 0$$



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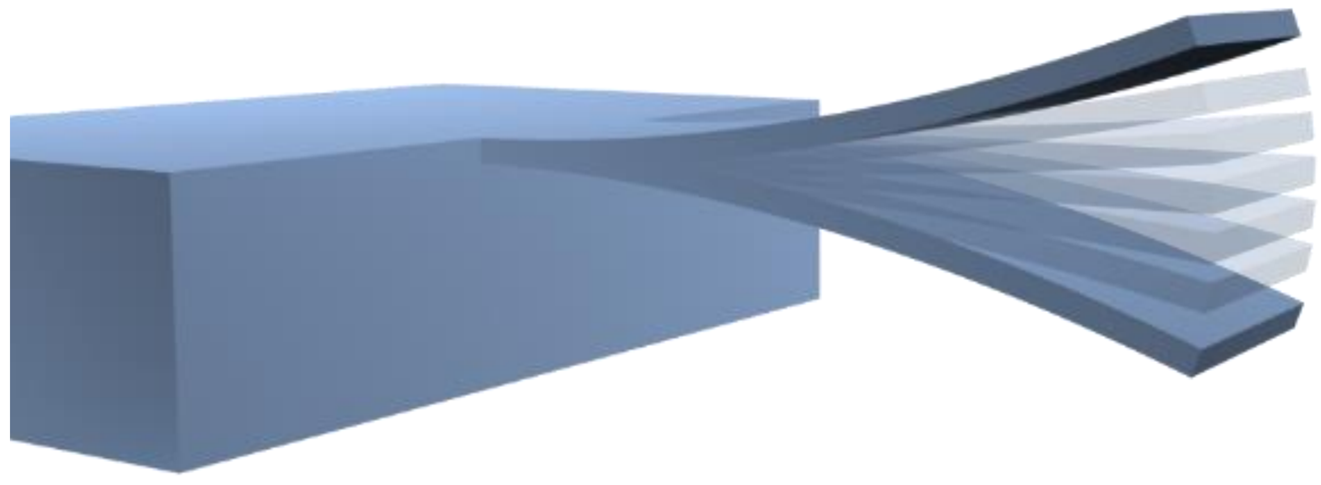

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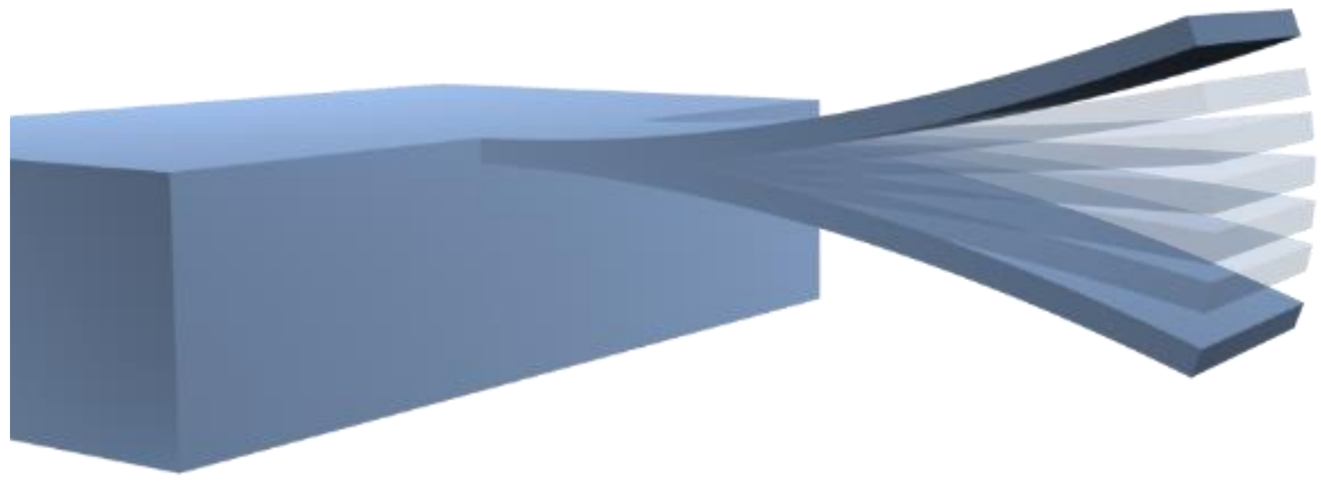

$$X(x = 0) = 0$$

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$$X(x) = C_1 \cos\left(\frac{x}{\lambda}\right) + C_2 \cosh\left(\frac{x}{\lambda}\right) + C_3 \sin\left(\frac{x}{\lambda}\right) + C_4 \sinh\left(\frac{x}{\lambda}\right)$$

$$\Big\downarrow\; X(x = 0) = 0$$

$$X(x) = C_1 \cos\left(\frac{x}{\lambda}\right) - C_1 \cosh\left(\frac{x}{\lambda}\right) + C_3 \sin\left(\frac{x}{\lambda}\right) + C_4 \sinh\left(\frac{x}{\lambda}\right)$$

$$\Big\downarrow\; X'(x = 0) = 0$$

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$$\Big\downarrow\; X''(x = L) = 0$$

$$X(x) = C_1 \left(\cos\left(\frac{x}{\lambda}\right) - \cosh\left(\frac{x}{\lambda}\right) + \frac{\cos\left(\frac{L}{\lambda}\right) + \cosh\left(\frac{L}{\lambda}\right)}{\sin\left(\frac{L}{\lambda}\right) + \sinh\left(\frac{L}{\lambda}\right)} \left(\sinh\left(\frac{x}{\lambda}\right) - \sin\left(\frac{x}{\lambda}\right)\right) \right)$$

Eigenvalues – Boundary conditions

- Solution at $t = 0$ is maximum:

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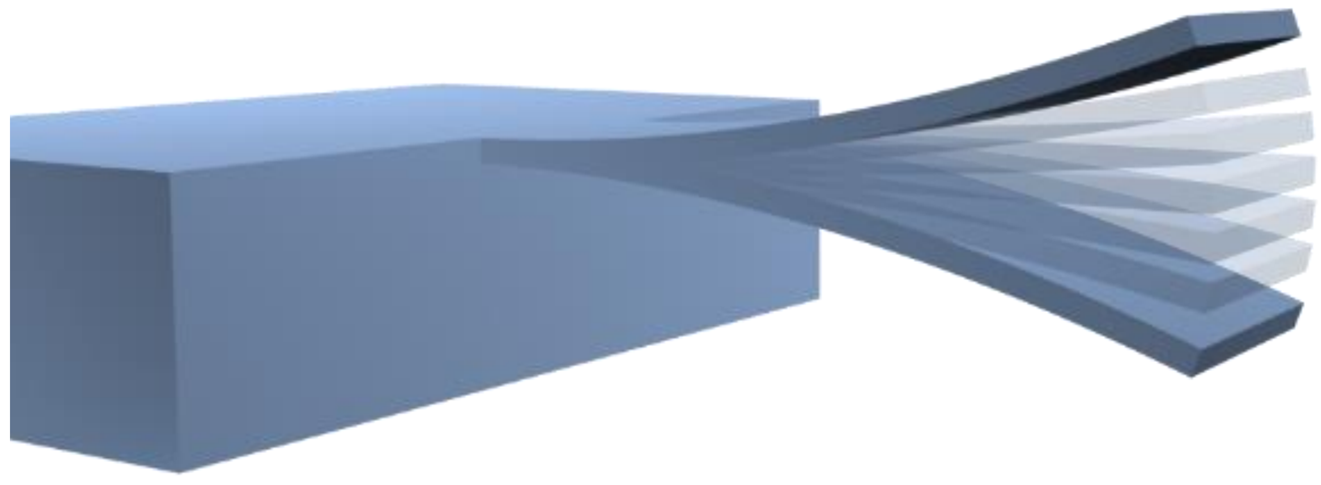

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$$X'''(x = L) = 0 \qquad C_1 = 0 \rightarrow Trivial\ solution\ - DISCARDED$$

$$\frac{2\left(1 + \cos\left(\frac{L}{\lambda}\right)\cosh\left(\frac{L}{\lambda}\right)\right)}{\lambda^3\left(\sin\left(\frac{L}{\lambda}\right) + \sinh\left(\frac{L}{\lambda}\right)\right)} = 0 \rightarrow 1 + \cos\left(\frac{L}{\lambda}\right)\cosh\left(\frac{L}{\lambda}\right) = 0$$

$$X(x = 0) = 0$$

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Eigenvalues – Boundary conditions

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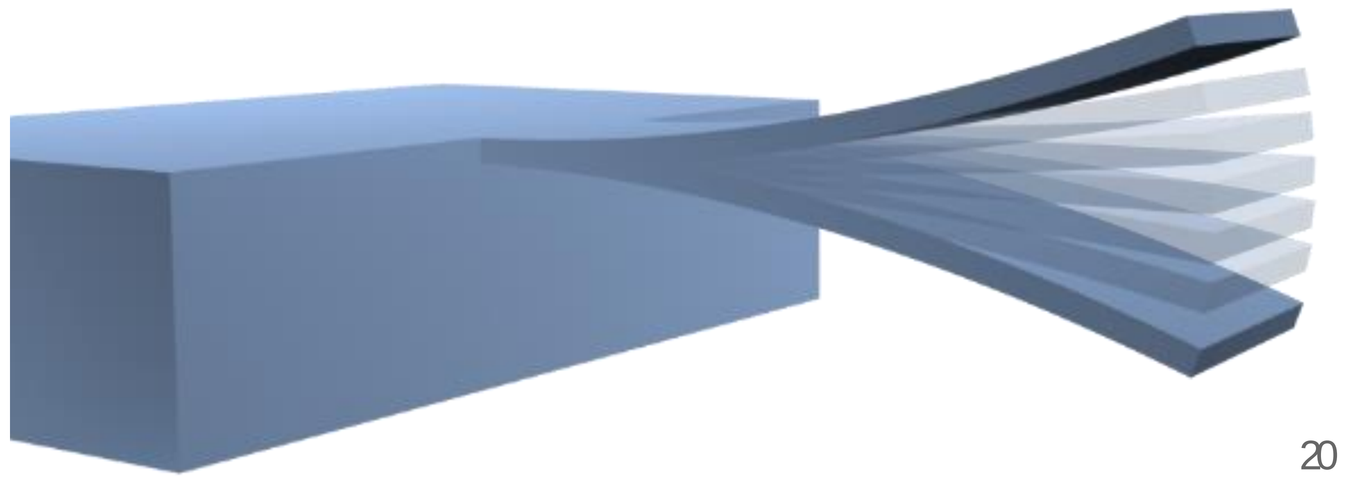
- Spatial boundary conditions:



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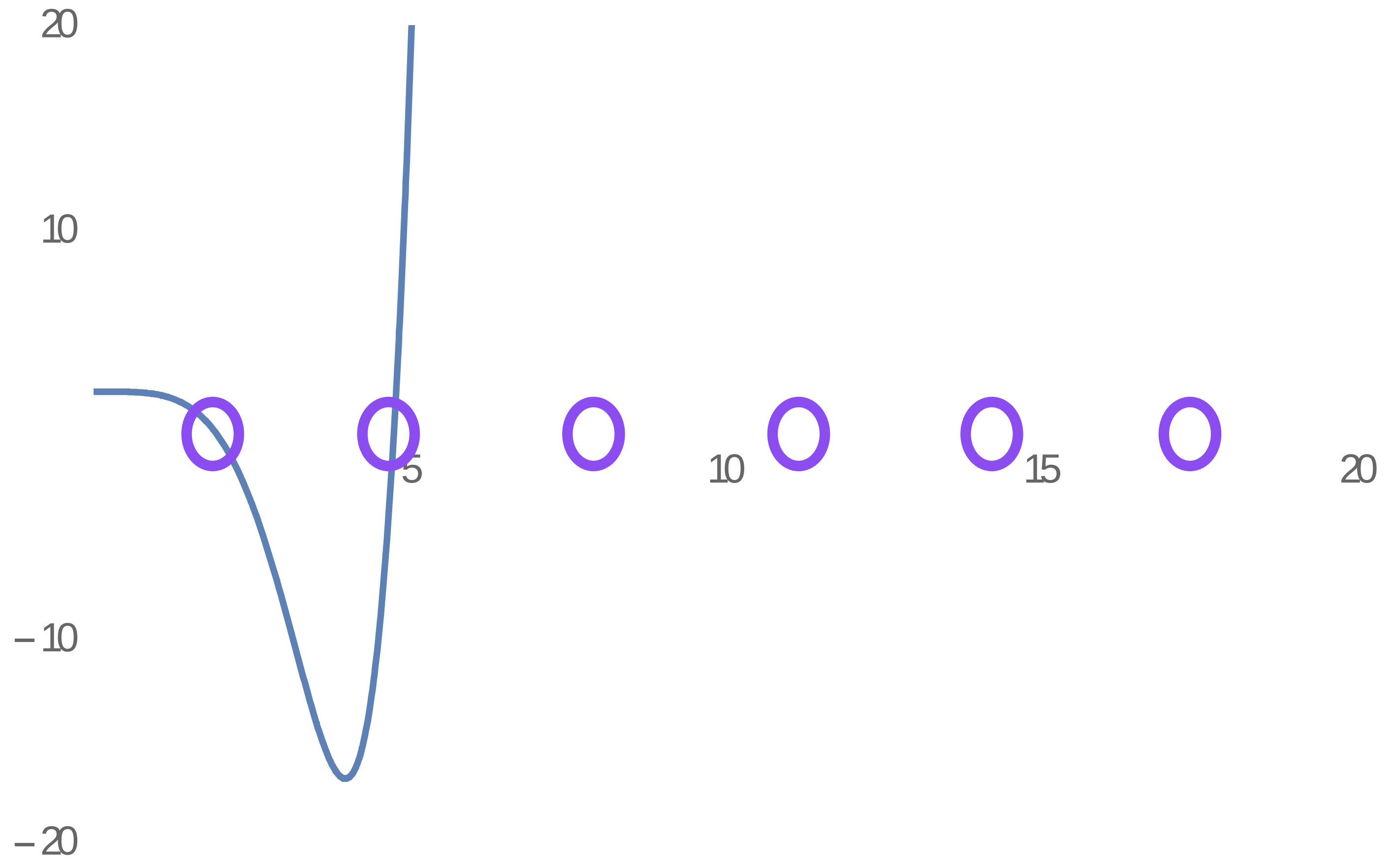

$\boldsymbol{n}$	$\boldsymbol{\frac{L}{\lambda_n}}$	$\boldsymbol{\left(\frac{L}{\lambda_n}\right)^2}$
1	1.875	3.515
2	4.694	22.03
3	7.855	61.7
≥ 4	$n\pi - \frac{pi}{2}$	$\left(n\pi - \frac{pi}{2}\right)^2$

Eigenvalues – Boundary conditions

- Solution at $t = 0$ is maximum:

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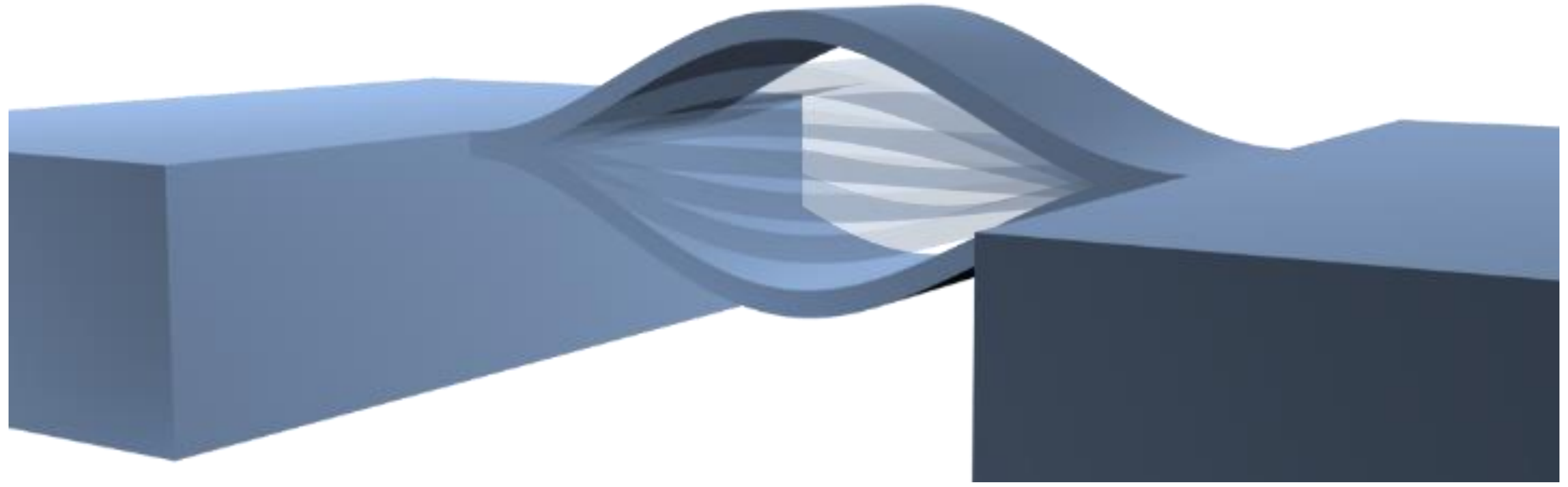

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$$X(x) = C_1\cos\left(\frac{x}{\lambda}\right) - C_1\cosh\left(\frac{x}{\lambda}\right) + C_3\sin\left(\frac{x}{\lambda}\right) + C_4\sinh\left(\frac{x}{\lambda}\right)$$

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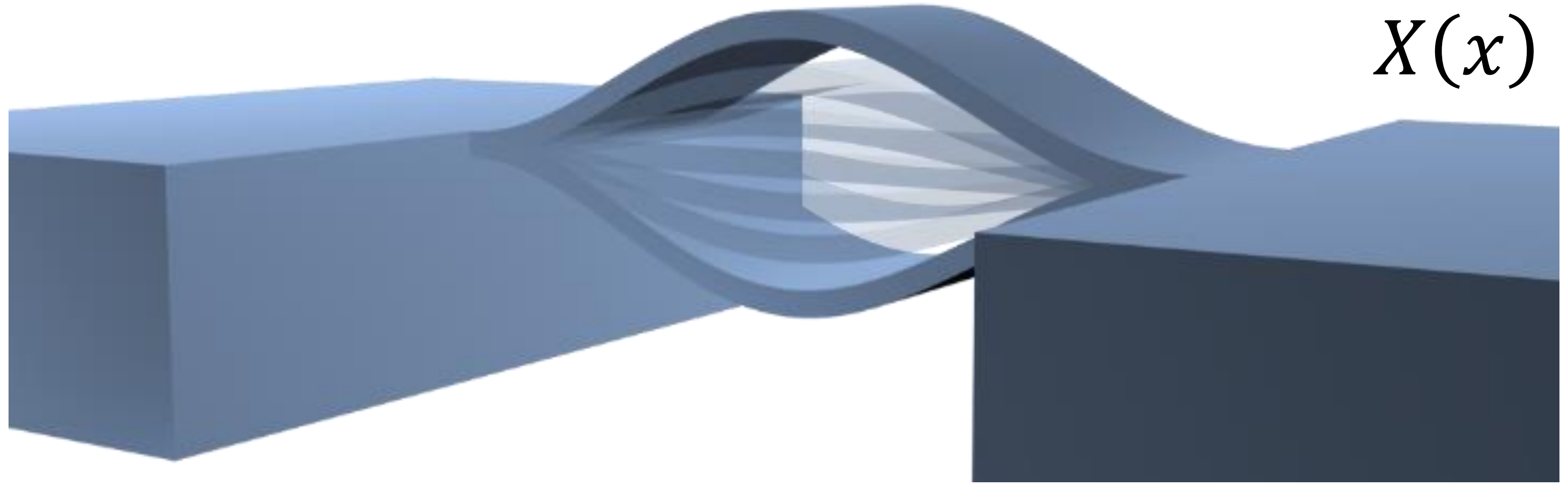
Eigenvalues – Boundary conditions

- Solution at $t = 0$ is maximum: $$T(t) = B_1 \cos(\omega t)\,;\ \boldsymbol{\omega^2} = \frac{\boldsymbol{EI}}{\boldsymbol{\rho A}}\frac{\mathbf{1}}{\boldsymbol{\lambda^4}}$$
- Spatial boundary conditions:

$$X(x) = C_1\left(\cos\left(\frac{x}{\lambda}\right) - \cosh\left(\frac{x}{\lambda}\right) + \frac{\sin\left(\frac{L}{\lambda}\right) + \sinh\left(\frac{L}{\lambda}\right)}{\cos\left(\frac{L}{\lambda}\right) - \cosh\left(\frac{L}{\lambda}\right)}\left(\sin\left(\frac{x}{\lambda}\right) - \sinh\left(\frac{x}{\lambda}\right)\right)\right)$$



$$X(x = L) = 0 \qquad C_1 = 0 \rightarrow Trivial\ solution\ - DISCARDED$$

$$\frac{2C_1\left(-1 + \cos\left(\frac{L}{\lambda}\right)\cosh\left(\frac{L}{\lambda}\right)\right)}{\cos\left(\frac{L}{\lambda}\right) - \cosh\left(\frac{L}{\lambda}\right)} = 0 \rightarrow -1 + \cos\left(\frac{L}{\lambda}\right)\cosh\left(\frac{L}{\lambda}\right) = 0$$

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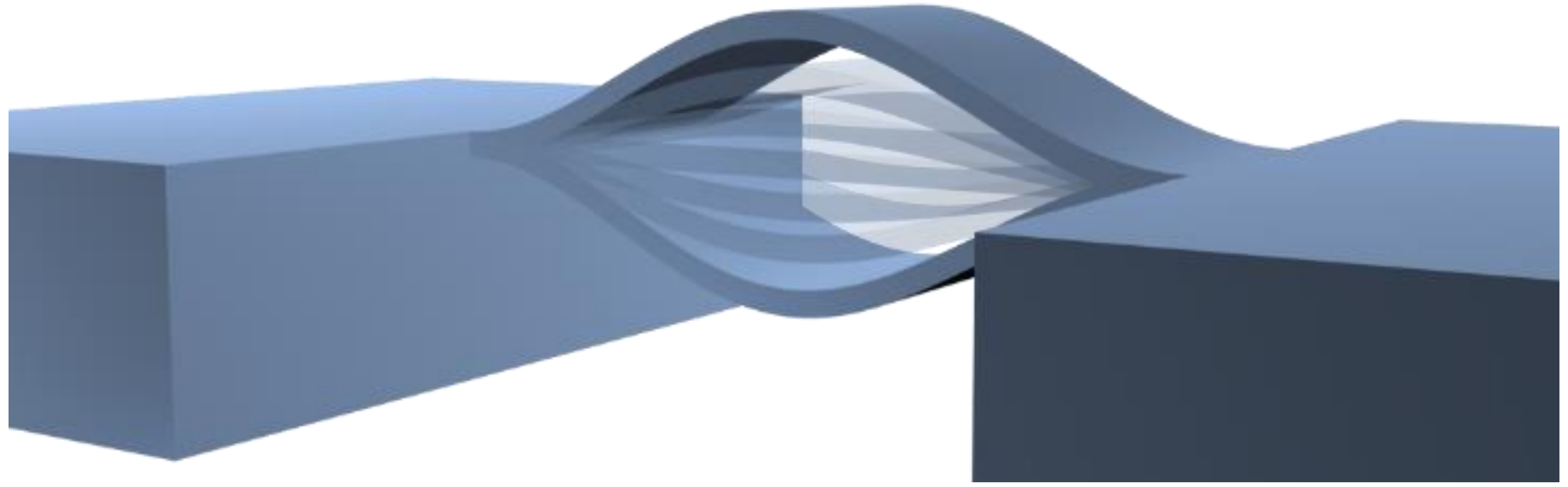

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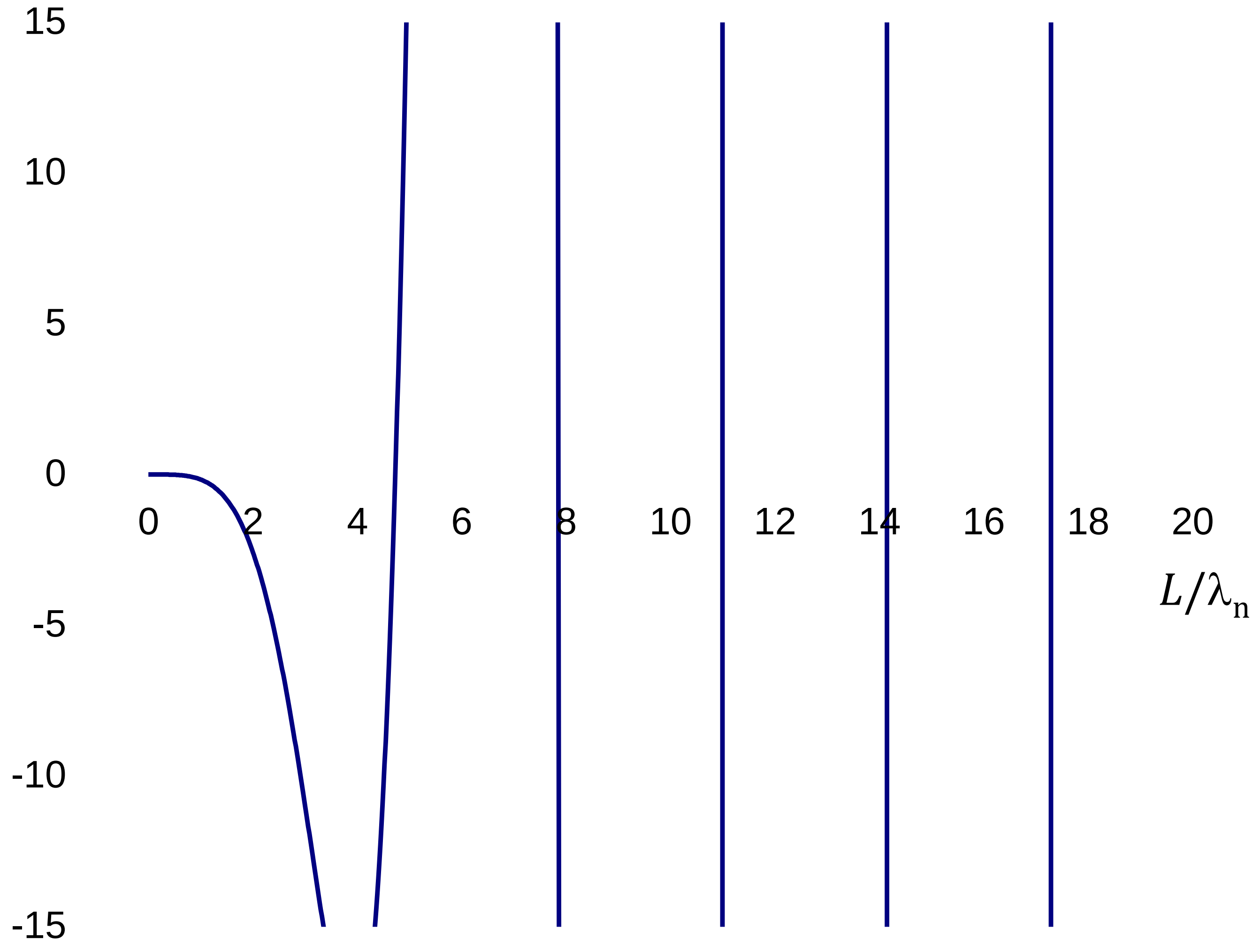
$$\cos\left(\frac{L}{\lambda}\right)\cosh\left(\frac{L}{\lambda}\right) - 1 = 0$$



$\boldsymbol{n}$	$\boldsymbol{\frac{L}{\lambda_n}}$	$\boldsymbol{\left(\frac{L}{\lambda_n}\right)^2}$
1	4.73	22.37
2	7.853	61.67
≥ 3	$n\pi + \frac{\pi}{2}$	$\left(n\pi + \frac{\pi}{2}\right)^2$

Eigenvectors

Eigenvectors // Eigenmodes

- Solution of the type: $w(x,t) = X(x)T(t)$
  - C-C beam
    - $w(x,t) = B_1 C_1 \cos(\omega_n t)\left(\cos\left(\frac{x}{\lambda_n}\right) - \cosh\left(\frac{x}{\lambda_n}\right) + \frac{\sin\left(\frac{L}{\lambda_n}\right)+\sinh\left(\frac{L}{\lambda_n}\right)}{\cos\left(\frac{L}{\lambda_n}\right)-\cosh\left(\frac{L}{\lambda_n}\right)}\left(\sin\left(\frac{x}{\lambda_n}\right) - \sinh\left(\frac{x}{\lambda_n}\right)\right)\right)$
  - Cantilever:
    - $w(x,t) = B_1 C_1 \cos(\omega_n t)\left(\cos\left(\frac{x}{\lambda_n}\right) - \cosh\left(\frac{x}{\lambda_n}\right) + \frac{\cos\left(\frac{L}{\lambda_n}\right)+\cosh\left(\frac{L}{\lambda_n}\right)}{\sin\left(\frac{L}{\lambda_n}\right)+\sinh\left(\frac{L}{\lambda_n}\right)}\left(\sinh\left(\frac{x}{\lambda_n}\right) - \sin\left(\frac{x}{\lambda_n}\right)\right)\right)$
  - In general: $w(x,t) \propto \cos(\omega_n t)\, \phi_n(x) \longrightarrow$ **Mode shape**

- What about $B_1 C_1$?
  - They are only partially important
  - They define what we can call the “normalization” of the modes

Why do we call them Eigenvectors?

- Remember that we were solving this equation

$$\frac{\partial^4 w}{\partial x^4} = -\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}$$

- We can now substitute: $w(x,t) \propto \cos(\omega_n t)\,\phi_n(x)$ and we end up with n equations:

$$\frac{EI}{\rho A}\frac{1}{\lambda_n^4}\phi_n(x) = \omega_n^2 \phi_n(x)$$

  - This is a diagonalized matrix problem, without differential terms, only algebraic

- It can be shown that the set of functions/vectors $\{\phi_1(x); \phi_2(x); \phi_3(x); \phi_4(x); \dots\}$ form an **orthogonal** base for the Hilbert space of functions within $x \in [0, L]$

- Why not **orthonormal**?

Normalization

- We can choose any normalization for the mode shapes that we want.
  - We need to be aware of the consequences and the things that will change once a choice is made

- Typical choices:

1
  - $\phi_n(x_{max}) = 1$

2
  - $\frac{1}{L}\int_0^L \phi_n(x)\,dx = 1 \leftrightarrow \int_0^1 \phi_n(\xi)\,d\xi = 1$

3
  - $\frac{1}{L}\int_0^L \phi_n^2(x)\,dx = 1 \leftrightarrow \int_0^1 \phi_n^2(\xi)\,d\xi = 1$
    - This one follows the definition of scalar product and allows to define: $\int_0^1 \phi_n(\xi)\phi_m(\xi)\,d\xi = \delta_{n,m}$

EPFL

6.2 – Resonators Lumped Model, Stress effect

ME-426 – Micro/Nanomechanical Devices

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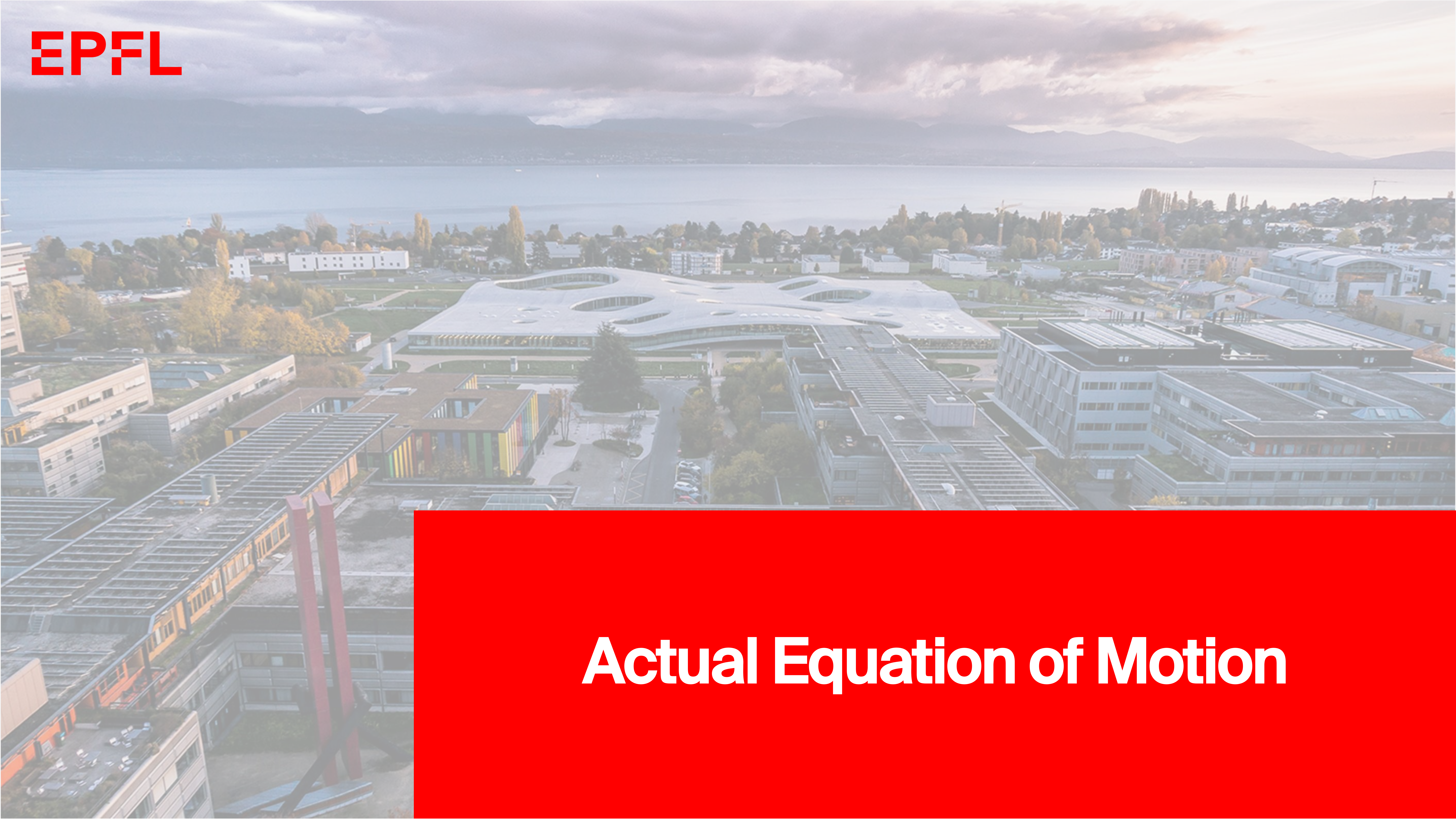
Introduction

- Equation of motion
- Lumped element model
- Stress effect on frequency
- Solution of the Equation of Motion



Actual Equation of Motion

Go back to the Eigenvalue equation...

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

Go back to the Eigenvalue equation…

$$\rho A \frac{\partial^2 w}{\partial t^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

- The general solution of this equation is more complicated than what we calculated before:

$$w(x,t) = \sum_{n=1}^{\infty} w_n \cos(\omega_n t)\, \phi_n(x) = \sum_{n=1}^{\infty} w_n(t) \phi_n(x)$$

  - Linear combination of the different vectors in the base – thus a general solution

- Remember:

$$\frac{1}{L}\int_0^L \phi_n(x)\phi_m(x)\, dx = \delta_{n,m} \quad \frac{1}{L}\int_0^L \phi_n^2(x)\, dx$$

$$\int_0^1 \phi_n(\xi)\phi_m(\xi)\, d\xi = \delta_{n,m} \quad \int_0^1 \phi_n^2(\xi)\, d\xi$$

$= 1$

3

...and now we write the actual equation

$$\rho A \frac{\partial^2 w}{\partial t^2} + c \frac{\partial w}{\partial t} + EI \frac{\partial^4 w}{\partial x^4} = p(x,t)$$

- Where c is the damping coefficient and $p(x,t)$ is the force density over differential length
- Even though the equation is different, whenever the damping is *small enough*, we can still use the same eigenvectors // normal modes that we calculated before, thus:

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- Even though the equation is different, whenever the damping is *small enough,* we can still use the same eigenvectors // normal modes that we calculated before, thus:

$$\rho A \sum_{n=1}^{\infty} \ddot{w}_n(t)\phi_n(x) + c\sum_{n=1}^{\infty} \dot{w}_n(t)\phi_n(x) + EI\sum_{n=1}^{\infty} w_n(t)\phi_n^{IV}(x) = p(x,t)$$

- In order to simplify this equation, we can check on what happens at one *mode*

Single mode solution

$$\rho A \sum_{j=1}^{\infty} \ddot{w}_j(t)\phi_j(x) + c \sum_{j=1}^{\infty} \dot{w}_j(t)\phi_j(x) + EI \sum_{j=1}^{\infty} w_j(t)\phi_j^{IV}(x) = p(x,t)$$

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- To do the projection we multiply everything times the mode shape $\phi_n(x)$ and integrate

$$\int_0^L \left(\rho A \sum_{j=1}^{\infty} \ddot{w}_j(t)\phi_j(x) + c \sum_{j=1}^{\infty} \dot{w}_j(t)\phi_j(x) + EI \sum_{j=1}^{\infty} w_j(t)\phi_j^{IV}(x) \right) \phi_n(x)\, dx = \int_0^L p(x,t)\phi_n(x)\, dx$$

$$\rho A \ddot{w}_n(t) \int_0^L \phi_n^2(x)\, dx + c\dot{w}_n(t) \int_0^L \phi_n^2(x)\, dx + EI w_n(t) \int_0^L {\phi_n''}^2(x)\, dx = \int_0^L p(x,t)\phi_m(x)\, dx$$

Single mode solution – Lump element model

$$\rho A \ddot{w}_n(t) \int_0^L \phi_n^2(x)\,dx + c\dot{w}_n(t) \int_0^L \phi_n^2(x)\,dx + EI w_n(t) \int_0^L \phi_n''^2(x)\,dx = \int_0^L p(x,t)\phi_m(x)\,dx$$

$$\updownarrow$$

$$\boldsymbol{m_{eff,n}\ddot{w}_n(t) + \Gamma_{eff,n}\dot{w}_n(t) + k_{eff,n}w_n(t) = F_{eff,n}(t)}$$

$$\boldsymbol{m\ddot{z}(t) + \Gamma\dot{z}(t) + kz(t) = F(t)}$$

Lumped model

- What does the variable $w_n(t)$ now stand for?
  - It depends on the normalization

1 $\phi_n(x_{max}) = 1 \qquad w(x,t) = \sum_{n=1}^{\infty} w_n \cos(\omega_n t)\,\phi_n(x) \rightarrow w_n(x_{max}, t) = w_n \cos(\omega_n t)\,\phi_n(x_{max}) = w_n \cos(\omega_n t)$

  - Lumped variable stands for the deflection of the point of maximum deflection

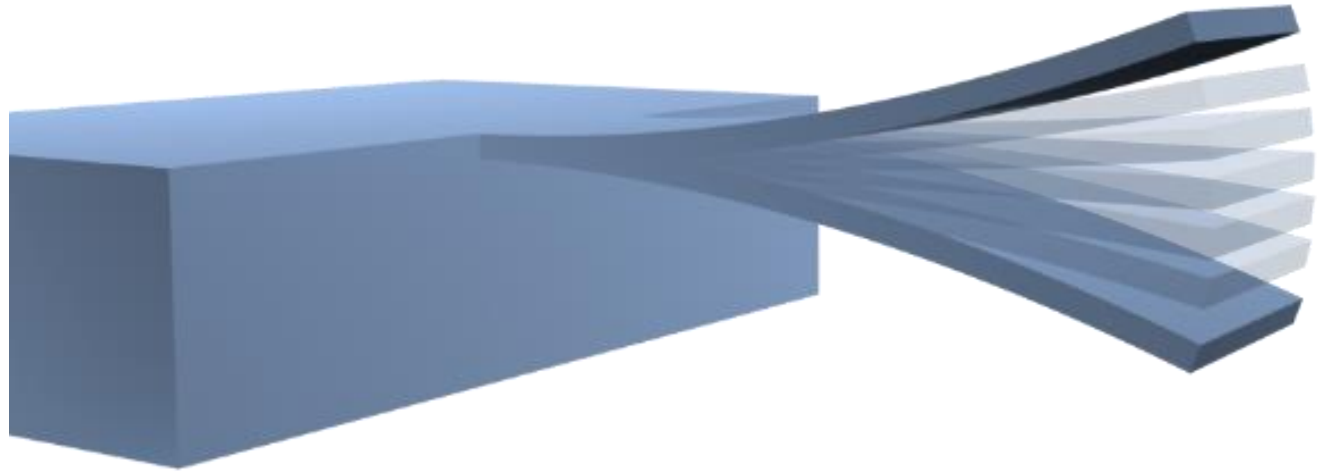
2 $\frac{1}{L}\int_0^L \phi_n(x)\,dx = 1 \leftrightarrow \int_0^1 \phi_n(\xi)\,d\xi = 1 \rightarrow \frac{1}{L}\int_0^L w_n(x,t)\,dx = \frac{1}{L}\int_0^L w_n \cos(\omega_n t)\,\phi_n(x_{max})\,dx = w_n \cos(\omega_n t)$

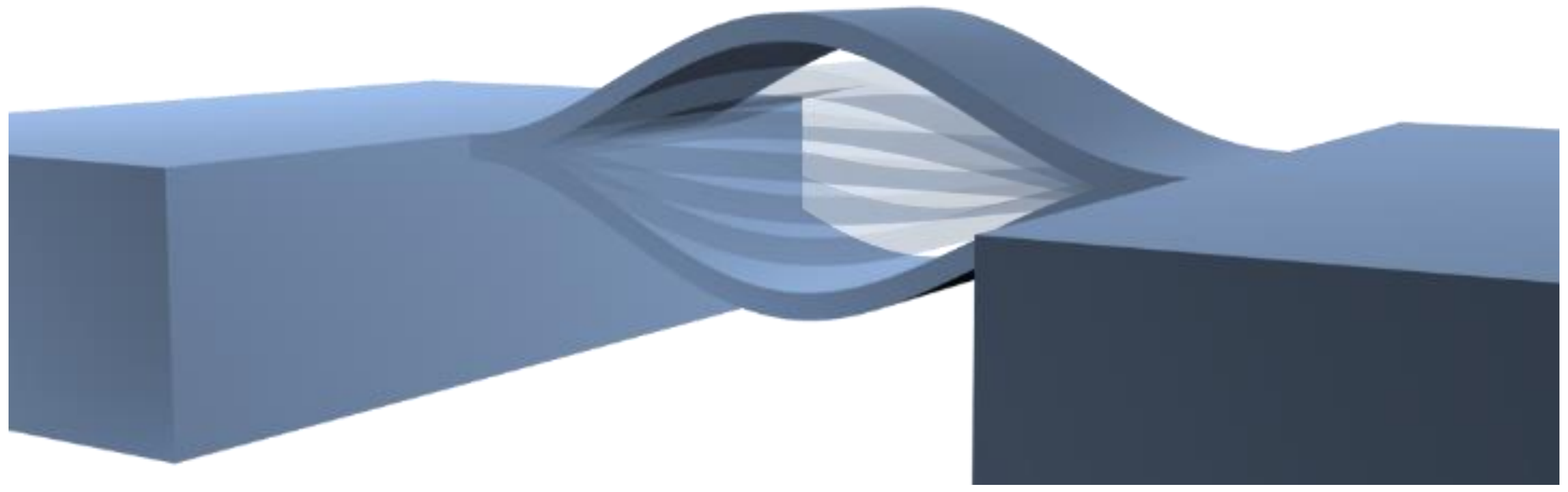
  - Lumped variable stands for the deflection of the *center of mass*, or *average deflection*

Effect of tension on frequency

Solving the Eigenvalue problem (II)

- We will consider flexural resonators





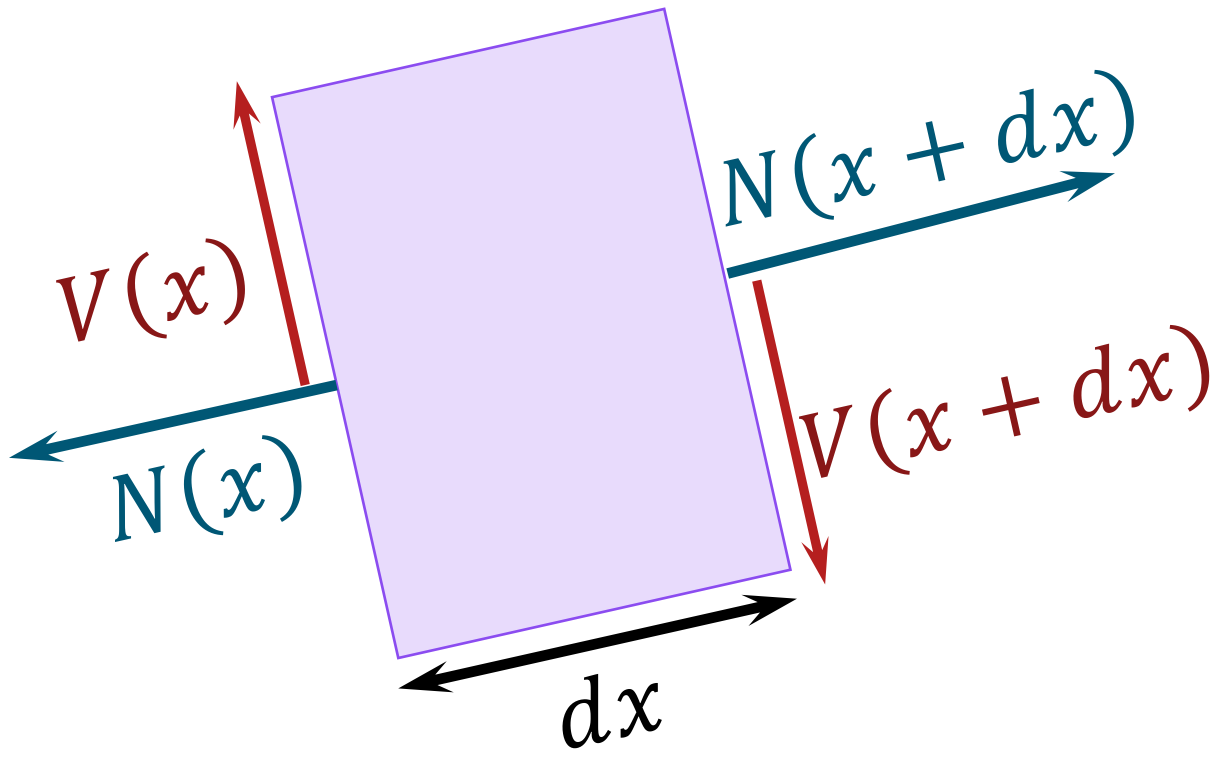
- We take a differential section, and we apply 2nd Newton's law:

$$F = ma \rightarrow V(x) - V(x+dx) + N(x+dx)w'(x+dx) - N(x)w'(x) = \rho A\, dx \frac{\partial^2 w}{\partial t^2}$$

$$-\frac{\partial V}{\partial x} dx + N(x)\frac{\partial^2 w}{\partial x^2} dx = \rho A\, dx \frac{\partial^2 w}{\partial t^2}$$

$$\frac{\partial V}{\partial x} - N\frac{\partial^2 w}{\partial x^2} = -\rho A \frac{\partial^2 w}{\partial t^2} \rightarrow \boldsymbol{\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right) - \sigma A \frac{\partial^2 w}{\partial x^2} = -\rho A \frac{\partial^2 w}{\partial t^2}}$$

$$\boldsymbol{\frac{\partial^4 w}{\partial x^4} - \frac{\sigma A}{EI}\frac{\partial^2 w}{\partial x^2} = -\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}}$$



Solving the Eigenvalue problem (II)

- We will consider flexural resonators, with uniform, monomaterial cross section

$$\frac{\partial^4 w}{\partial x^4} - \frac{\sigma A}{EI}\frac{\partial^2 w}{\partial x^2} = -\frac{\rho A}{EI}\frac{\partial^2 w}{\partial t^2}$$

- Still, we can do separation of variables, solution of the form: $w(x,t) = X(x)T(t)$

- But this equation, in general, is much harder to solve

- We can do one simple thing, when the effect of stress is relatively small:
  - Assume that the eigenvectors are not modified
  - Look how much the eigenvalues change

Solving the Eigenvalue problem (II)

$$\rho A \frac{\partial^2 w}{\partial t^2} - \sigma A \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

- We project on one *mode*

Solving the Eigenvalue problem (II)

$$\rho A \frac{\partial^2 w}{\partial t^2} - \sigma A \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

- We project on one *mode*

Solving the Eigenvalue problem (II)

$$\rho A \frac{\partial^2 w}{\partial t^2} - \sigma A \frac{\partial^2 w}{\partial x^2} + EI \frac{\partial^4 w}{\partial x^4} = 0$$

- We project on one *mode*

$$\rho A \ddot{w}_n(t) \int_0^L \phi_n^2(x)\, dx + EI w_n(t) \int_0^L {\phi_n''}^2(x)\, dx + \sigma A w_n(t) \int_0^L {\phi_n'}^2(x)\, dx = 0$$

$$\boldsymbol{\omega_n^2 = \frac{EI}{\rho A} \frac{\int_0^L {\phi_n''}^2(x)\, dx}{\int_0^L \phi_n^2(x)\, dx} + \frac{\sigma A}{\rho A} \frac{\int_0^L {\phi_n'}^2(x)\, dx}{\int_0^L \phi_n^2(x)\, dx}}$$

$$\boldsymbol{\omega_n^2 = \frac{EI}{\rho A} \frac{1}{L^4} \frac{\int_0^1 {\phi_n''}^2(\xi)\, d\xi}{\int_0^1 \phi_n^2(\xi)\, d\xi} + \frac{\sigma A}{\rho A} \frac{1}{L^2} \frac{\int_0^1 {\phi_n'}^2(\xi)\, d\xi}{\int_0^1 \phi_n^2(\xi)\, d\xi}}$$

Solving the Eigenvalue problem (II)

$$\omega_n^2 = \frac{EI}{\rho A}\frac{1}{L^4}\frac{\int_0^1 \phi_n''^2(\xi)\,d\xi}{\int_0^1 \phi_n^2(\xi)\,d\xi} + \frac{\sigma A}{\rho A}\frac{1}{L^2}\frac{\int_0^1 \phi_n'^2(\xi)\,d\xi}{\int_0^1 \phi_n^2(\xi)\,d\xi}$$

$$\omega_n^2 = \frac{EI}{\rho A}\frac{1}{L^4}\frac{\int_0^1 \phi_n''^2(\xi)\,d\xi}{\int_0^1 \phi_n^2(\xi)\,d\xi}\left(1 + \frac{\sigma A}{EI}L^2\frac{\int_0^1 \phi_n'^2(\xi)\,d\xi}{\int_0^1 \phi_n''^2(\xi)\,d\xi}\right)$$

$$\omega_n^2 = \frac{EI}{\rho A}\frac{1}{L^4}\beta_n^2\left(1 + \frac{\sigma A}{EI}L^2\gamma_n\right)$$

$$\omega_n = \sqrt{\frac{EI}{\rho A}}\frac{1}{L^2}\beta_n\left(1 + \frac{\sigma A}{2EI}L^2\gamma_n\right) = \sqrt{\frac{E}{\rho}}\frac{h}{L^2}\frac{\beta_n}{2\sqrt{3}}\left(1 + 6\frac{\sigma}{E}\left(\frac{L}{h}\right)^2\gamma_n\right)$$

Solving the E.O.M.

Solving the E.O.M. – Fourier transform

$$m_{eff,n}\ddot{w}_n(t) + \Gamma_{eff,n}\dot{w}_n(t) + k_{eff,n}w_n(t) = F_{eff,n}(t)$$

$$m_{eff,n}\ddot{w}_n(t) + \frac{m_{eff,n}\omega_n}{Q_n}\dot{w}_n(t) + k_{eff,n}w_n(t) = F_{eff,n}(t)$$

$$w_n(\omega) = \frac{\frac{F_n(\omega)}{m_{eff,n}}}{\omega_n^2 - \omega^2 + i\,\frac{\omega_n\omega}{Q_n}}$$

Solving the E.O.M. – Fourier transform

$$m_{eff,n}\ddot{w}_n(t) + \frac{m_{eff,n}\omega_n}{Q_n}\dot{w}_n(t) + k_{eff,n}w_n(t) = F_{eff,n}(t)$$

$$w_n(\omega) = \frac{\frac{F_n(\omega)}{m_{eff,n}}}{\omega_n^2 - \omega^2 + i\frac{\omega_n\omega}{Q_n}} \rightarrow |w_n(\omega)|^2 = \frac{\left|\frac{F_n(\omega)}{m_{eff,n}}\right|^2}{(\omega_n^2-\omega^2)^2 + \left(\frac{\omega_n\omega}{Q_n}\right)^2}$$

$$\approx \frac{\left|\frac{F_n(\omega)}{2m_{eff,n}}\right|^2}{\left(1-\frac{\omega}{\omega_n}\right)^2 + \left(\frac{1}{2Q_n}\right)^2}$$

Solving the E.O.M. – What if we have stress???

$$m_{eff,n}\ddot{w}_n(t) + \frac{m_{eff,n}\omega_n}{Q_n}\dot{w}_n(t) + k^{\sigma}_{eff,n}w_n(t) = F_{eff,n}(t)$$

$$m_{eff,n} = \rho A L \int_0^1 \phi_n^2(\xi)\, d\xi$$

$$Q_n^{-1} = \frac{c}{\rho A \omega_n}$$

$$k^{\sigma=0}_{eff,n} = \frac{EI}{L^3}\int_0^1 \phi_n''^2(\xi)\, d\xi$$

$$F_{eff,n} = \int_0^L p(x,t)\phi_n(x)\, dx$$

$$\boldsymbol{k^{\sigma}_{eff,n} = \frac{EI}{L^3}\int_0^1 \phi_n''^2(\xi)\, d\xi + \frac{\sigma A}{L}\int_0^1 \phi_n'^2(\xi)\, d\xi}$$

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6.3 – Resonators Coupled resonators, Nonlinearity

ME-426 – Micro/Nanomechanical Devices

Prof. Guillermo Villanueva

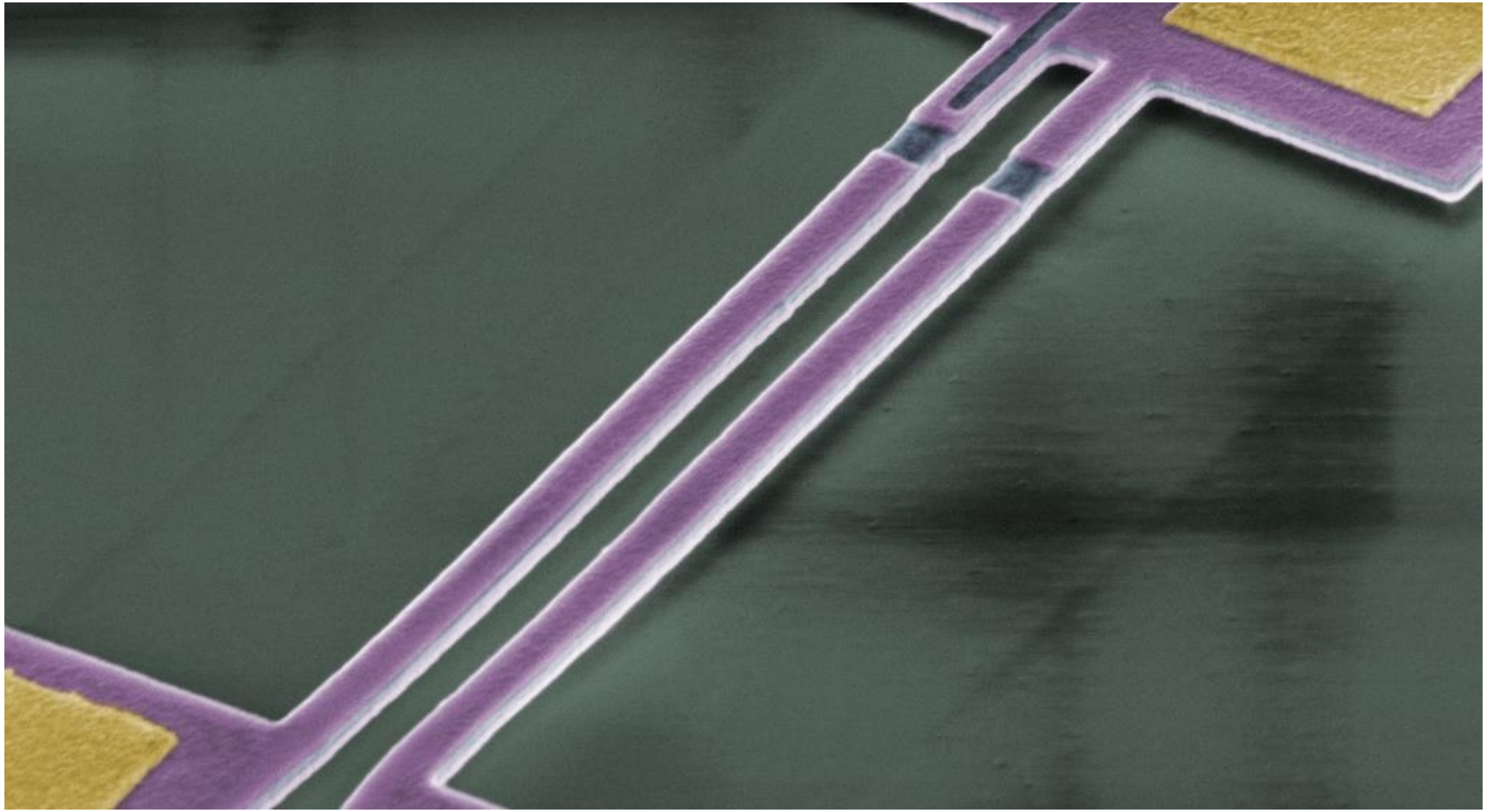
École polytechnique fédérale de Lausanne

Introduction

- Coupled resonators
- Nonlinearity - Origins
- Dynamic nonlinearity – Duffing

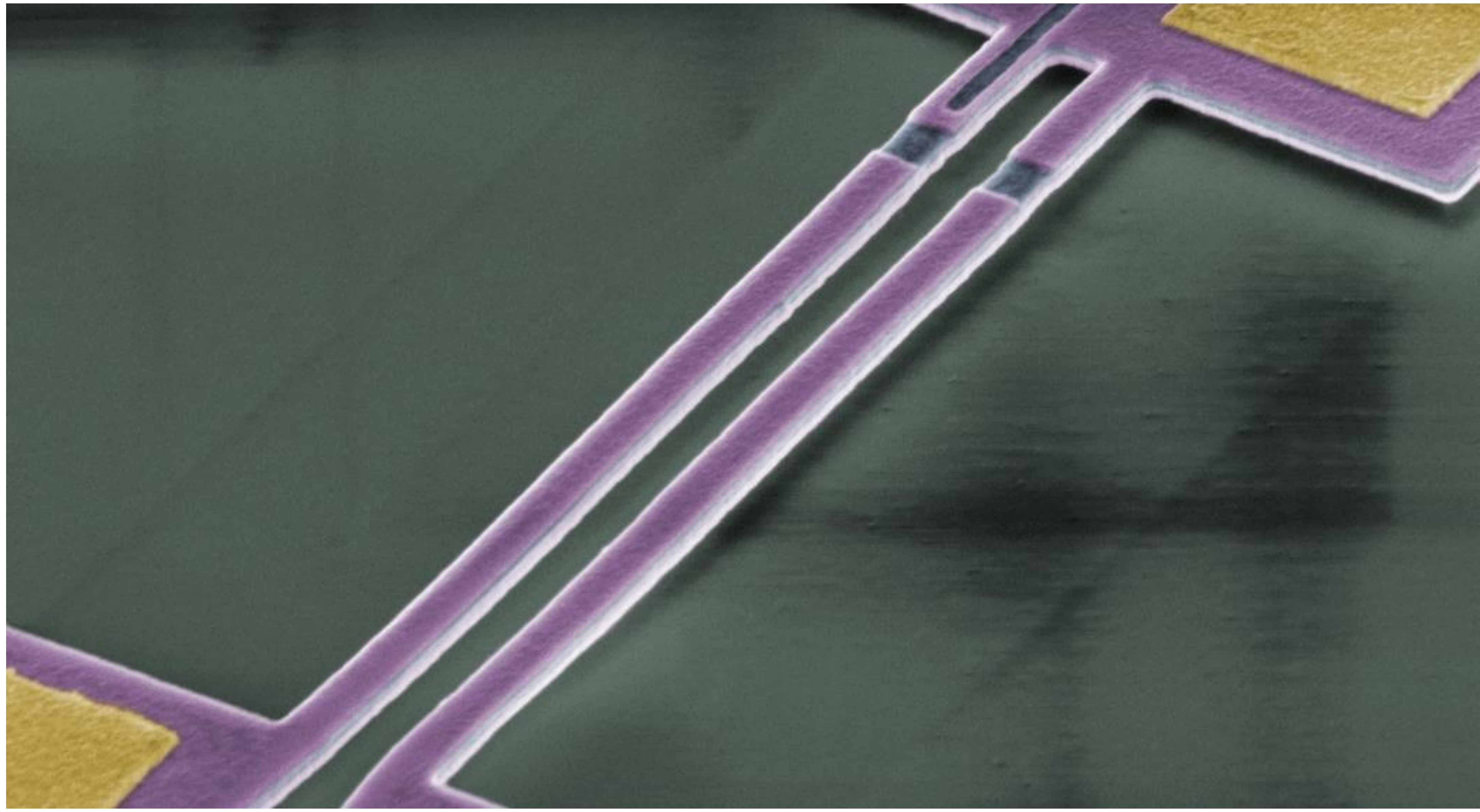
Coupled resonators

Coupled devices



Coupled devices

- Individual resonators
  - Coupled via shared mechanical ledge

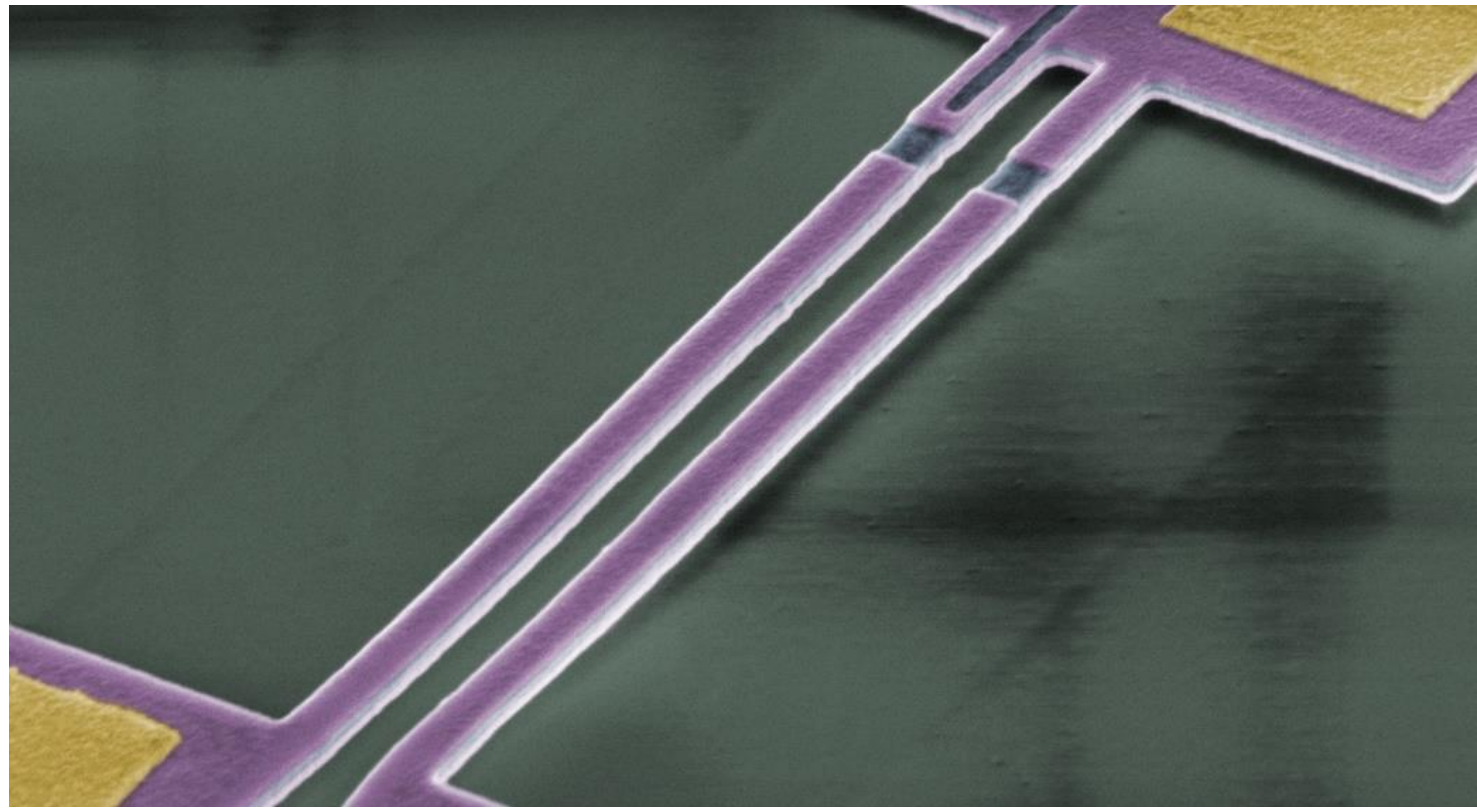


- Modelling
  - Coupling: $F_{coupling} = \chi(x_1 - x_2)$

- Equations of motion
  - $\ddot{x}_1 + \omega_1 Q^{-1}\dot{x}_1 + \omega_1^2 x_1 + D(x_1 - x_2) = F_1$
  - $\ddot{x}_2 + \omega_2 Q^{-1}\dot{x}_2 + \omega_2^2 x_2 - D(x_1 - x_2) = F_2$

Eigenvalues//Eigenmodes

- We solve by forcing both resonators to move with the same frequency
  - Analogous to diagonalize the system



  - $-\omega^2 x_1 + \frac{j\omega\omega_1}{Q} x_1 + \omega_1^2 x_1 + D(x_1 - x_2) = F_1$
  - $-\omega^2 x_2 + \frac{j\omega\omega_2}{Q} x_2 + \omega_2^2 x_2 - D(x_1 - x_2) = F_2$
  - $\begin{pmatrix} -\omega^2 + \frac{j\omega\omega_1}{Q} + \omega_1^2 + D & -D \\ -D & -\omega^2 + \frac{j\omega\omega_2}{Q} + \omega_2^2 + D \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$

- Eigenvalues
  - $\omega_I^2 = \omega_{av}^2 + D\left(1 - \sqrt{1 + \left(\frac{\delta}{D}\right)^2}\right); \omega_{II}^2 = \omega_{av}^2 + D\left(1 + \sqrt{1 + \left(\frac{\delta}{D}\right)^2}\right)$
  - with: $\omega_{av}^2 = \frac{\omega_1^2 + \omega_2^2}{2}$; $\delta = \frac{\omega_2^2 - \omega_1^2}{2}$

Eigenvalues//Eigenmodes

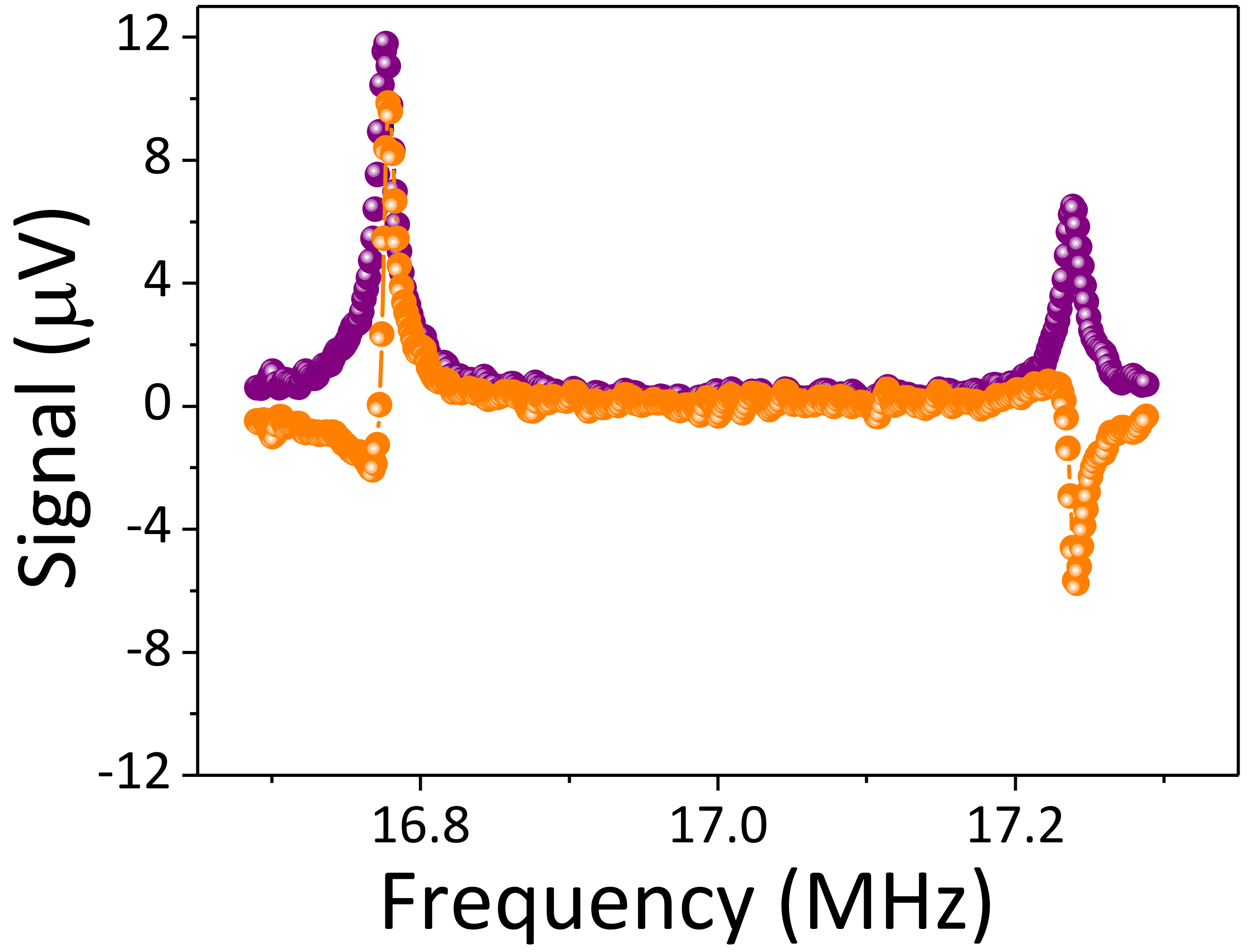
- Eigenvalues
  - $\omega_I^2 = \omega_{av}^2 + D\left(1 - \sqrt{1 + \left(\frac{\delta}{D}\right)^2}\right); \omega_{II}^2 = \omega_{av}^2 + D\left(1 + \sqrt{1 + \left(\frac{\delta}{D}\right)^2}\right)$
  - with: $\omega_{av}^2 = \frac{\omega_1^2 + \omega_2^2}{2}$; $\delta = \frac{\omega_2^2 - \omega_1^2}{2}$

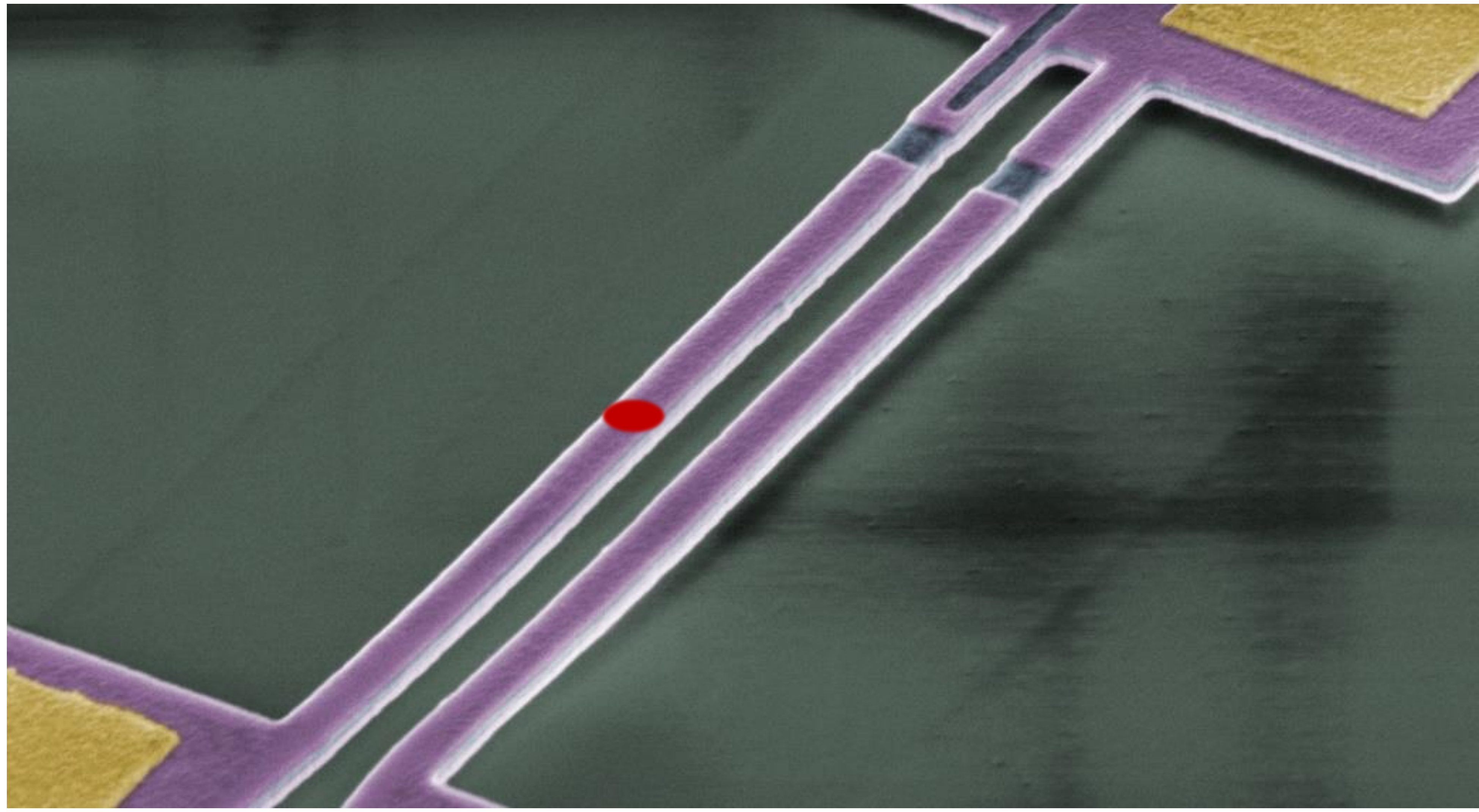
- Eigenvectors
  - $e_I = \frac{1}{N_I}\begin{pmatrix} \frac{\delta + \sqrt{\delta^2 + D^2}}{D} \\ 1 \end{pmatrix}$; $e_{II} = \frac{1}{N_{II}}\begin{pmatrix} \frac{\delta - \sqrt{\delta^2 + D^2}}{D} \\ 1 \end{pmatrix}$

- Note that for no initial mismatch ($\omega_1 = \omega_2$)
  - $\omega_I^2 = \omega_m^2$; $e_I = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}$
  - $\omega_{II}^2 = \omega_m^2 + 2D$; $e_{II} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$

Eigenvalues//Eigenvectors

- Experimental measurement
- Eigenvalues
  - $\frac{\omega_I}{2\pi} = 16.79\ \mathrm{MHz};\ \frac{\omega_{II}}{2\pi} = 17.25\ \mathrm{MHz}$
- Eigenvectors
  - $e_I = \begin{pmatrix} 1 \\ 0.767 \end{pmatrix}; \quad e_{II} = \begin{pmatrix} 1 \\ -1.25 \end{pmatrix}$





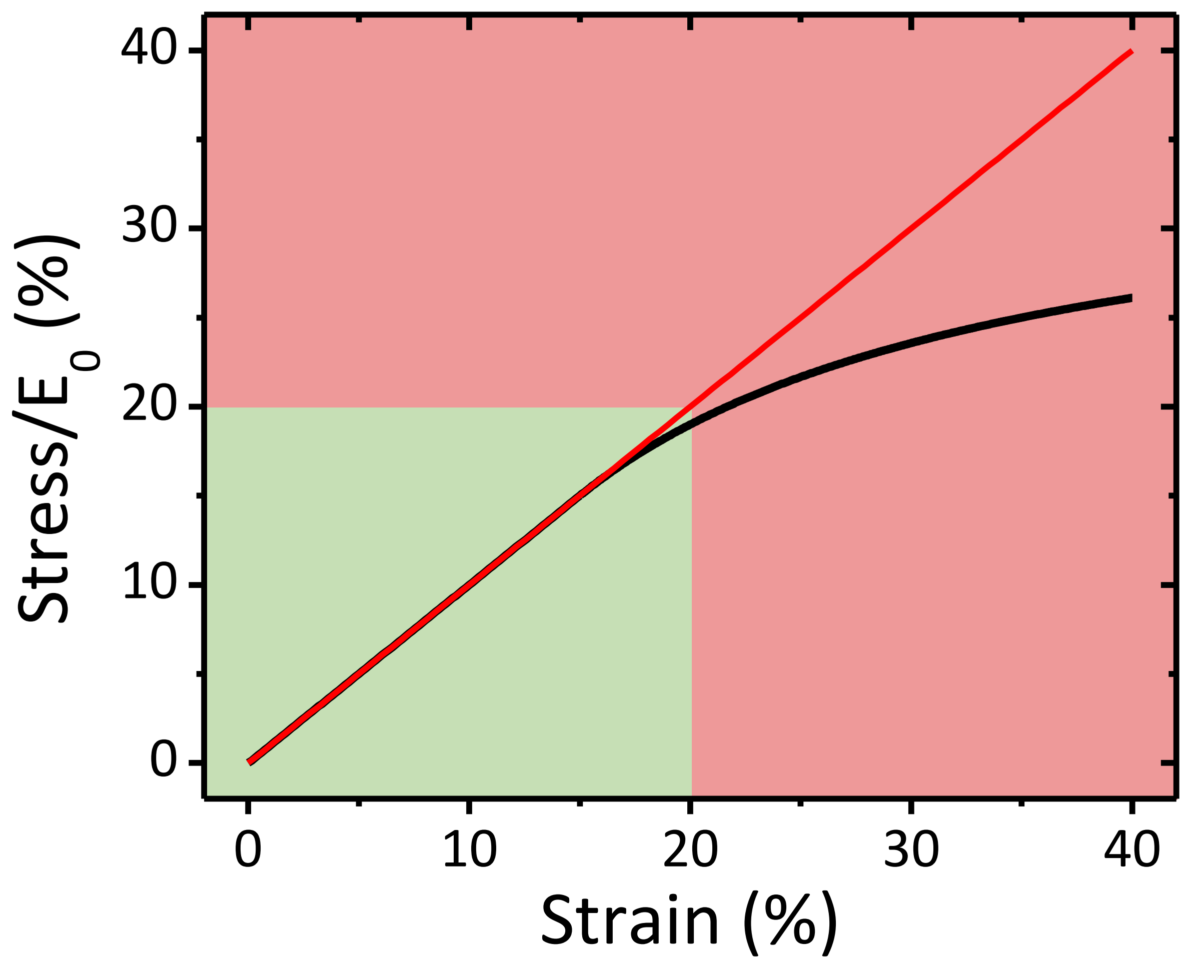
Nonlinearity

Where does NL come from?

- Material nonlinearity
  - e.g. Strain-Stress ratio change

$$\sigma = E_0\varepsilon + E_1\varepsilon^2 + E_2\varepsilon^3 + \cdots$$

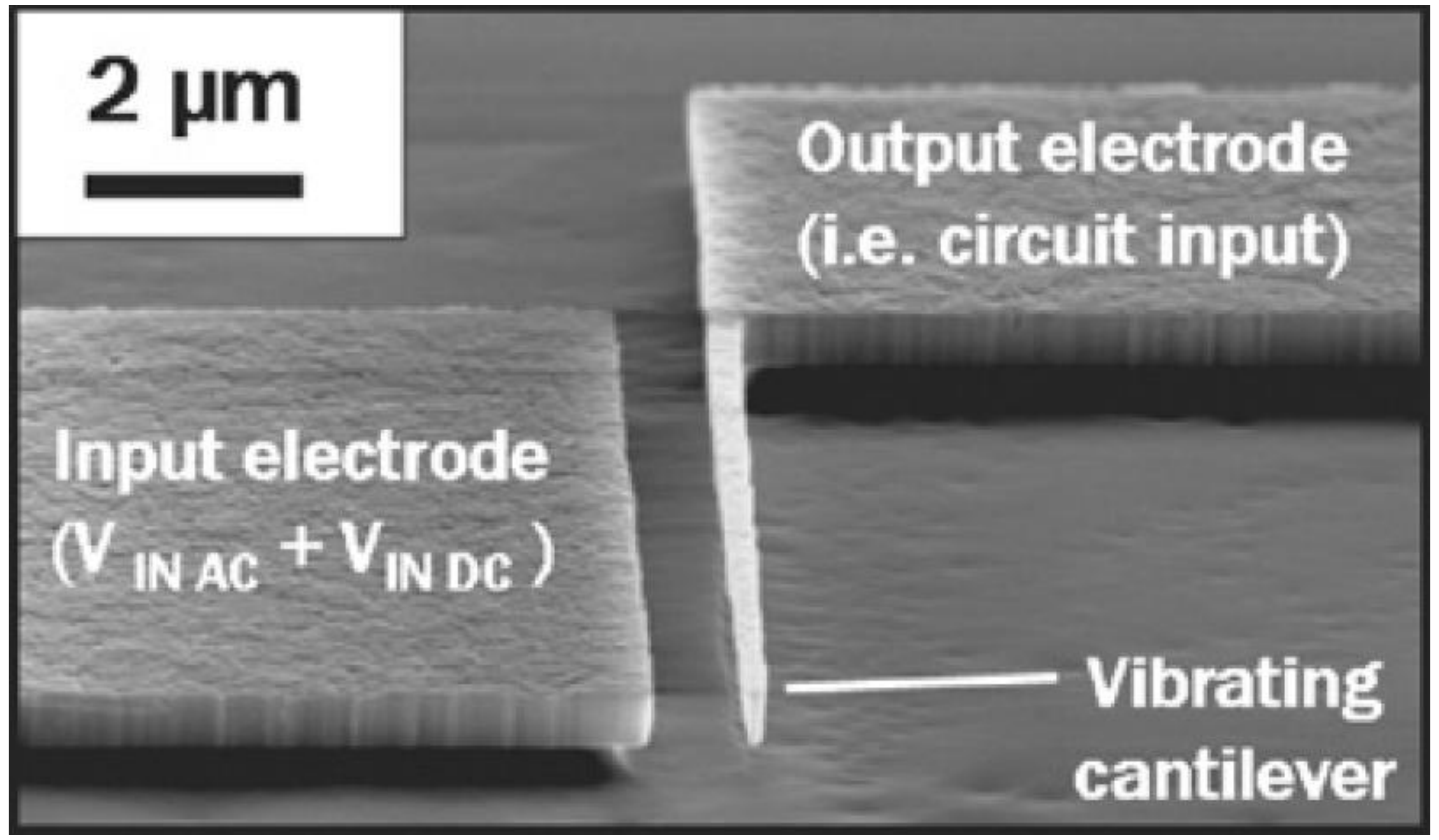
$$F = k_0 x + k_1 x^2 + k_2 x^3 + \cdots$$



Origins of nonlinearity

- Material nonlinearity
  - e.g. Strain-Stress ratio change
- Actuation nonlinearity
  - e.g. capacitive drive

$$F = -kx - \frac{c}{(g_0 + x)^2} \approx \left(-k + \frac{c_2}{g_0^3}\right)x + \frac{c_3}{g_0^4}x^2 + \frac{c_4}{g_0^5}x^3 + \cdots$$



Kacem et al., JMM, 2010

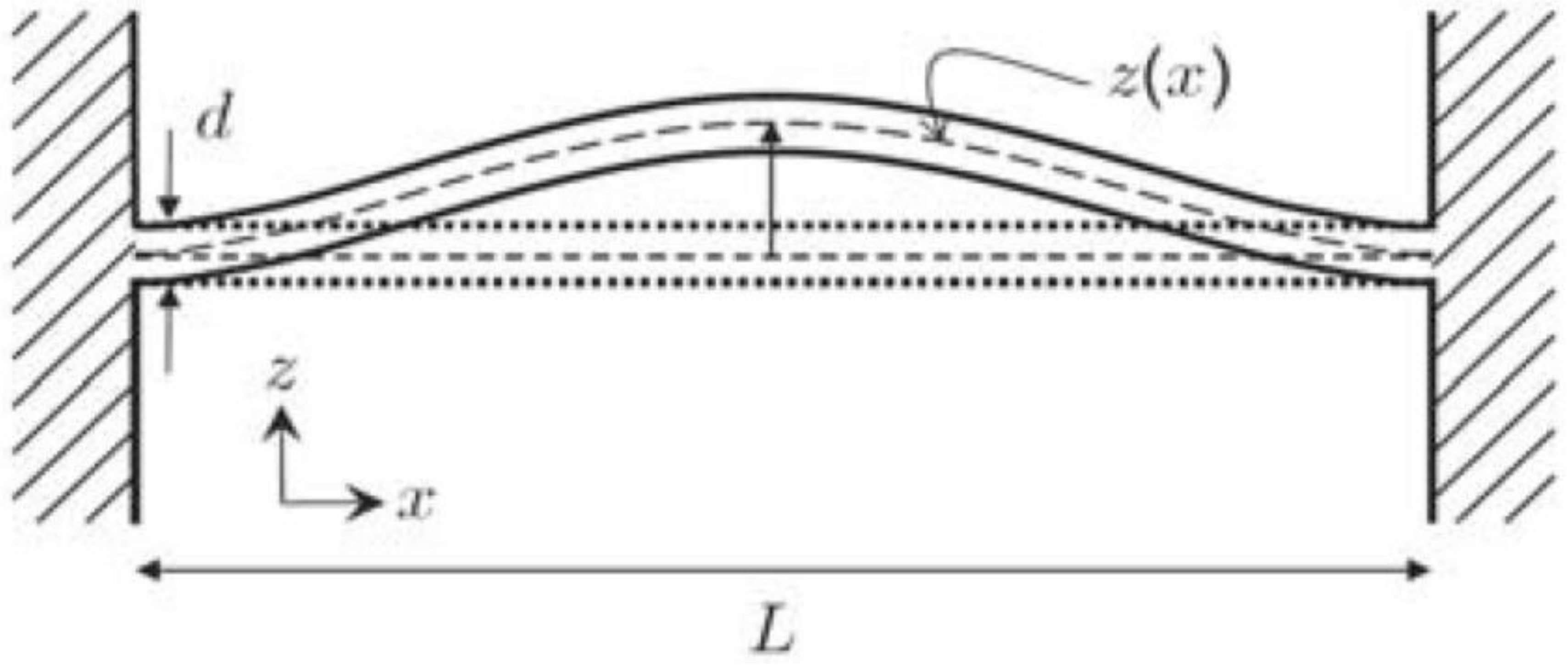
Origins of nonlinearity

- Material nonlinearity
  - e.g. Strain-Stress ratio change
- Actuation nonlinearity
  - e.g. capacitive drive
- "Large displacements" nonlinearity
  - e.g. departure from small deformations approximation

$$\varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial x}\right)^2\right]$$

Origins of nonlinearity

- Material nonlinearity
  - e.g. Strain-Stress ratio change
- Actuation nonlinearity
  - e.g. capacitive drive



- "Large displacements" nonlinearity
  - e.g. Built-in tension from deformation
  - Deflection of the beam gives rise to an elongation caused by the motion:

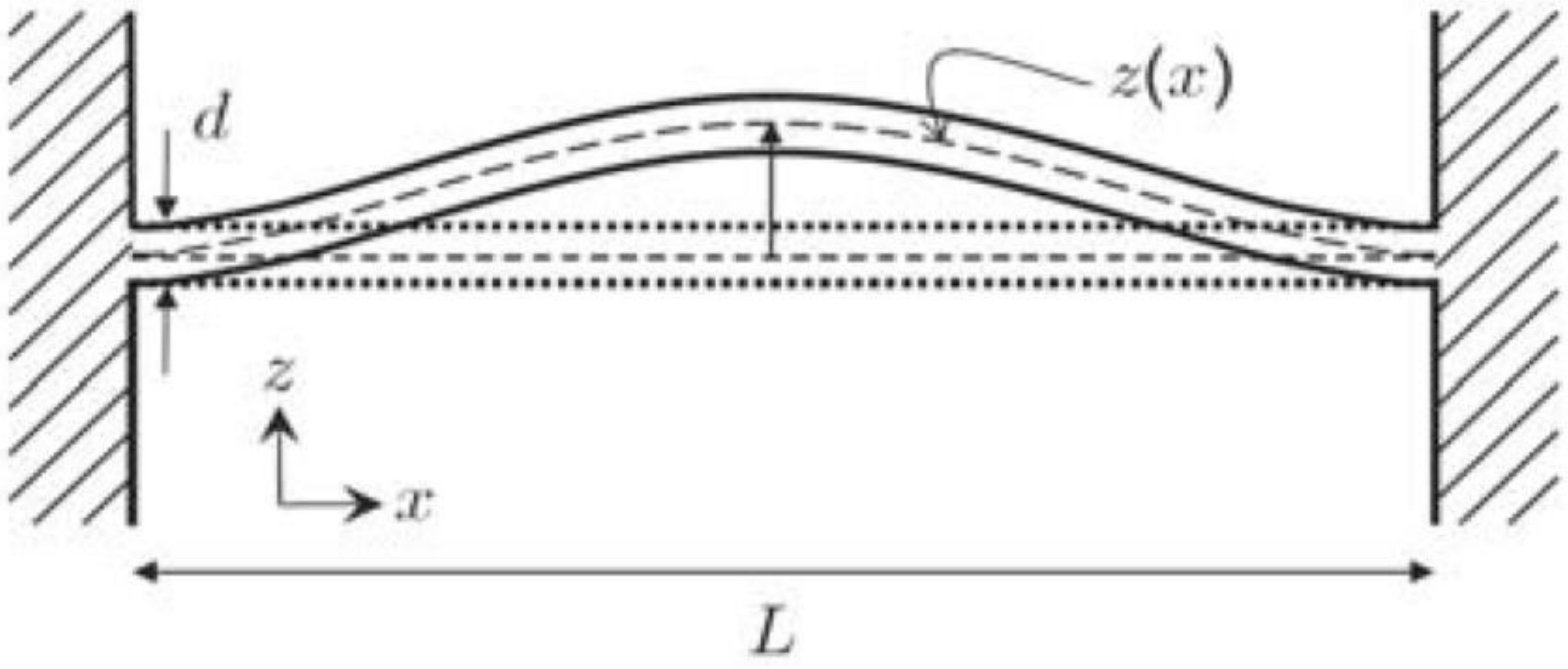
$$\frac{\Delta L}{L} = \varepsilon_x = \frac{1}{2L}\int_0^L w'(x)^2 dx$$

  - This required elongation causes a stress to appear in the beam

$$\sigma_x = E\varepsilon_x \rightarrow N = E\varepsilon_x A$$

Built-in tension Nonlinearity

- $N_{induced} = \frac{EA}{2L}\int_0^L w'(x)^2 \, \mathrm{d}x$



Built-in tension Nonlinearity

- $N_{induced} = \frac{EA}{2L}\int_0^L w'(x)^2\,\mathrm{d}x$

$$\rho A\frac{\partial^2 w}{\partial t^2} - \left(\sigma_0 + \frac{E}{2L}\int_0^L w'(x)^2\,\mathrm{d}x\right)A\frac{\partial^2 w}{\partial x^2} + EI\frac{\partial^4 w}{\partial x^4} = 0$$

$$m_{eff,n}\ddot{w}_n(t) + \frac{m_{eff,n}\omega_n}{Q_n}\dot{w}_n(t) + k_{eff,n}^{\sigma_0}w_n(t) + \boldsymbol{\alpha w_n^3(t)} = F_{eff,n}(t)$$

Lumped parameters - Normalization

$$m_{eff,n}\ddot{w}_n(t) + \Gamma_{eff,n}\dot{w}_n(t) + k^{\sigma}_{eff,n}w_n(t) + \alpha_{eff,n}w_n^3(t) = F_{eff,n}(t)$$

$$m_{eff,n} = \rho A L \int_0^1 \phi_n^2(\xi)\, d\xi$$

$$\Gamma_{eff,n} = cL \int_0^1 \phi_n^2(\xi)\, d\xi$$

$$k^{\sigma}_{eff,n} = \frac{EI}{L^3}\int_0^1 \phi_n''^2(\xi)\, d\xi + \frac{\sigma A}{L}\int_0^1 \phi_n'^2(\xi)\, d\xi$$

$$\alpha_{eff,n} = \frac{EA}{2L^3}\left(\int_0^1 \phi_n'^2(\xi)\, d\xi\right)^2$$

$$F_{eff,n} = L\int_0^1 p(x,t)\phi_n(\xi)\, d\xi$$

Dynamic Nonlinearity

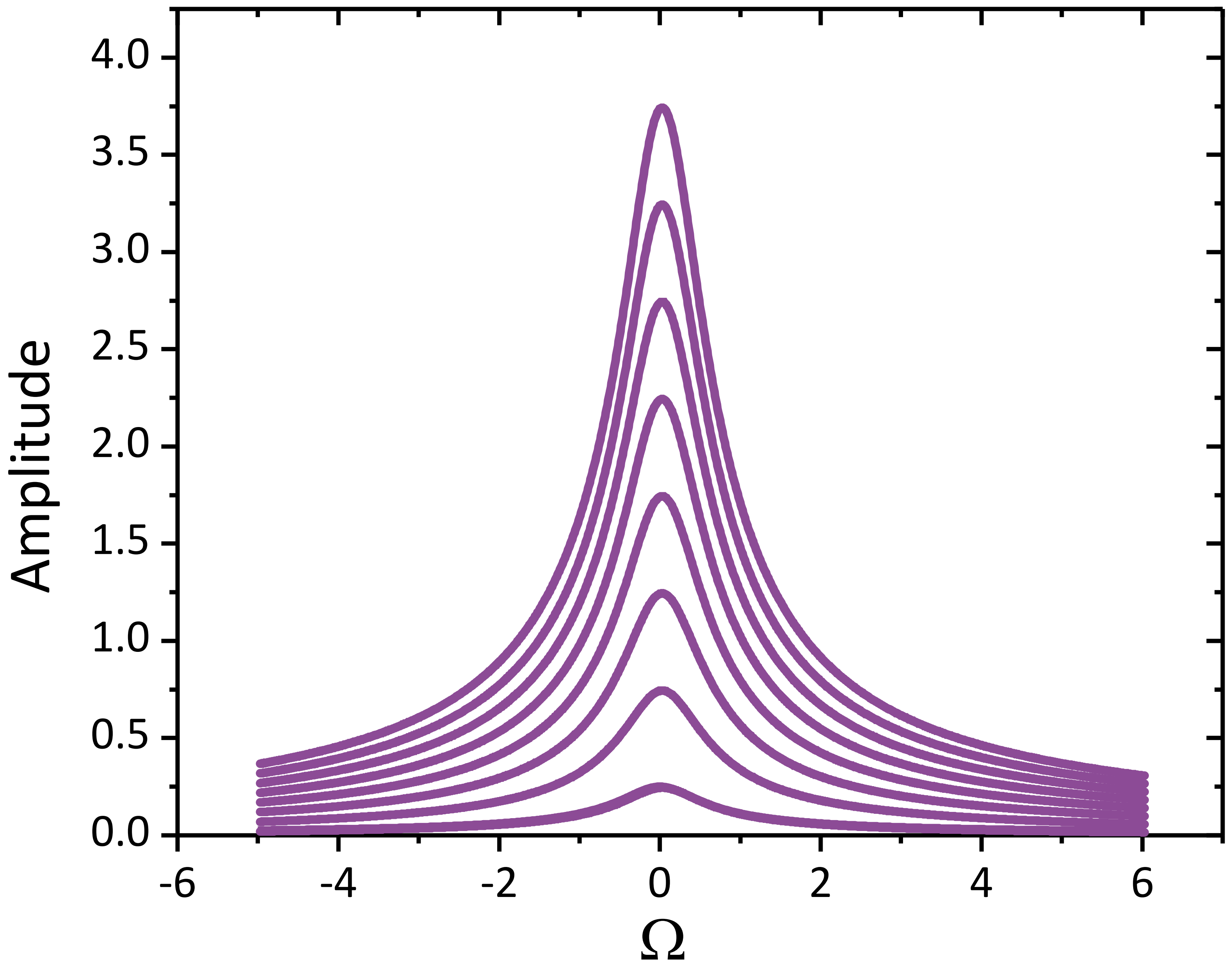
How does NL come into our E.O.M.?

- Linear: $$m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$$

  - Solution: $z(t) = z_0 \cdot \cos(\omega t - \varphi)$

  or

  - Solution: $z(t) = \hat{z_0} \cdot e^{j\omega t} + c.c.$



How does NL come into our E.O.M.?

- Linear: $m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$
- Nonlinear: $m\ddot{z} + \Gamma\dot{z} + kz + \alpha z^3 + \eta z^2\dot{z} = G\cos(\omega t)$

  - Lifshitz & Cross, ch. 1, Review of Nonlinear Dynamics, 2008
  - Multiple scales and secular perturbation theory: $z(t) \propto A(t) \cdot e^{i\omega t} + c.c.$
  - Change of variables:
    - Slow time: $T = t\frac{1}{Q}\sqrt{\frac{k}{m}} = t\epsilon\omega_0$; $\Omega = \frac{\omega - \omega_0}{\omega_0}Q$; $A = |z|\left(\frac{\alpha\, Q}{m\,\omega_0}\right)^{\frac{1}{2}} e^{i\Omega T}$

How does NL come into our E.O.M.?

- Linear: $m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$
- Nonlinear: $m\ddot{z} + \Gamma\dot{z} + kz + \alpha z^3 + \eta z^2\dot{z} = G\cos(\omega t)$

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  - Multiple scales and secular perturbation theory: $z(t) \propto A(t) \cdot e^{i\omega t} + c.c.$
  - Change of variables:
    - Slow time: $T = t\frac{1}{Q}\sqrt{\frac{k}{m}} = t\epsilon\omega_0;$ $\quad \Omega = \frac{\omega-\omega_0}{\omega_0}Q;$ $\quad A = |z|\left(\frac{\alpha\, Q}{m\,\omega_0}\right)^{\frac{1}{2}} e^{i\Omega T}$
  - Amplitude equation:

$$\frac{dA}{dT} + \frac{1}{2}A - \frac{1}{8}(3i - \eta)|A|^2 A = -i\frac{g}{2}e^{\mathrm{i}\Omega_\mathrm{D}\mathrm{T}}$$

How does NL come into our E.O.M.?

- Linear: $m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$
- Nonlinear: $m\ddot{z} + \Gamma\dot{z} + kz + \alpha z^3 + \eta z^2\dot{z} = G\cos(\omega t)$

  - Lifshitz & Cross, ch. 1, Review of Nonlinear Dynamics, 2008
  - Multiple scales and secular perturbation theory: $z(t) \propto A(t) \cdot e^{i\omega t} + c.c.$
  - Amplitude equation:

$$\frac{dA}{dT} + \frac{1}{2}A - \frac{1}{8}(3i - \eta)|A|^2 A = -i\frac{g}{2}e^{i\Omega_D T}$$

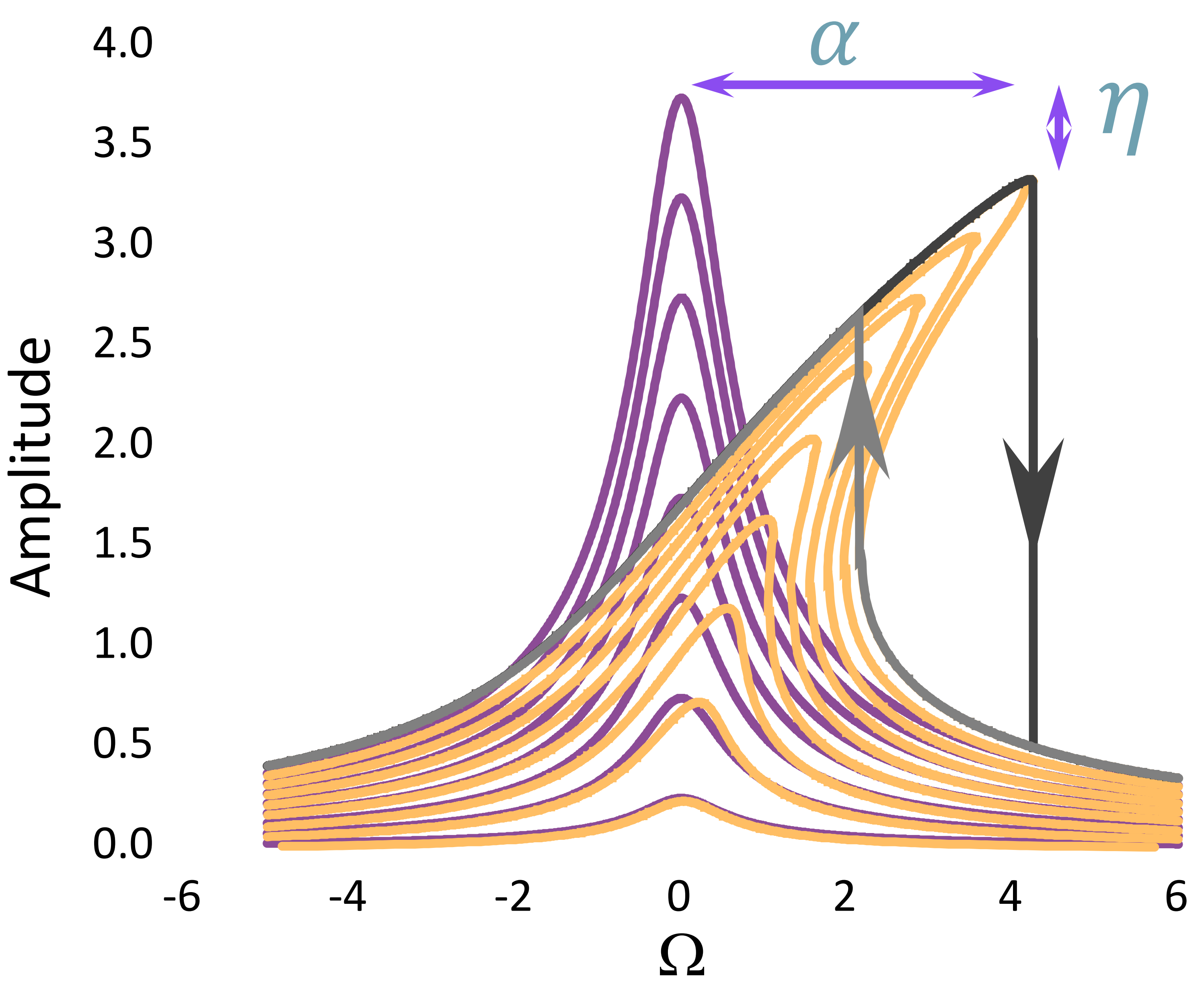
  - Back to dimensional magnitudes:

$$|z|^2 = \frac{\left(\frac{G}{2m\omega_0^2}\right)^2}{\left(\frac{\omega - \omega_0}{\omega_0} - \frac{3}{8}\frac{\alpha}{m\omega_0^2}|z|^2\right)^2 + \left(\frac{1}{2Q} + \frac{1}{8}\frac{\eta}{m\omega_0^2}|z|^2\right)^2};\qquad \tan(\phi) = \frac{\frac{1}{2Q} + \frac{1}{8}\frac{\eta}{m\omega_0^2}|z|^2}{\frac{\omega - \omega_0}{\omega_0} - \frac{3}{8}\frac{\alpha}{m\omega_0^2}|z|^2}$$

How does NL come into our E.O.M.?

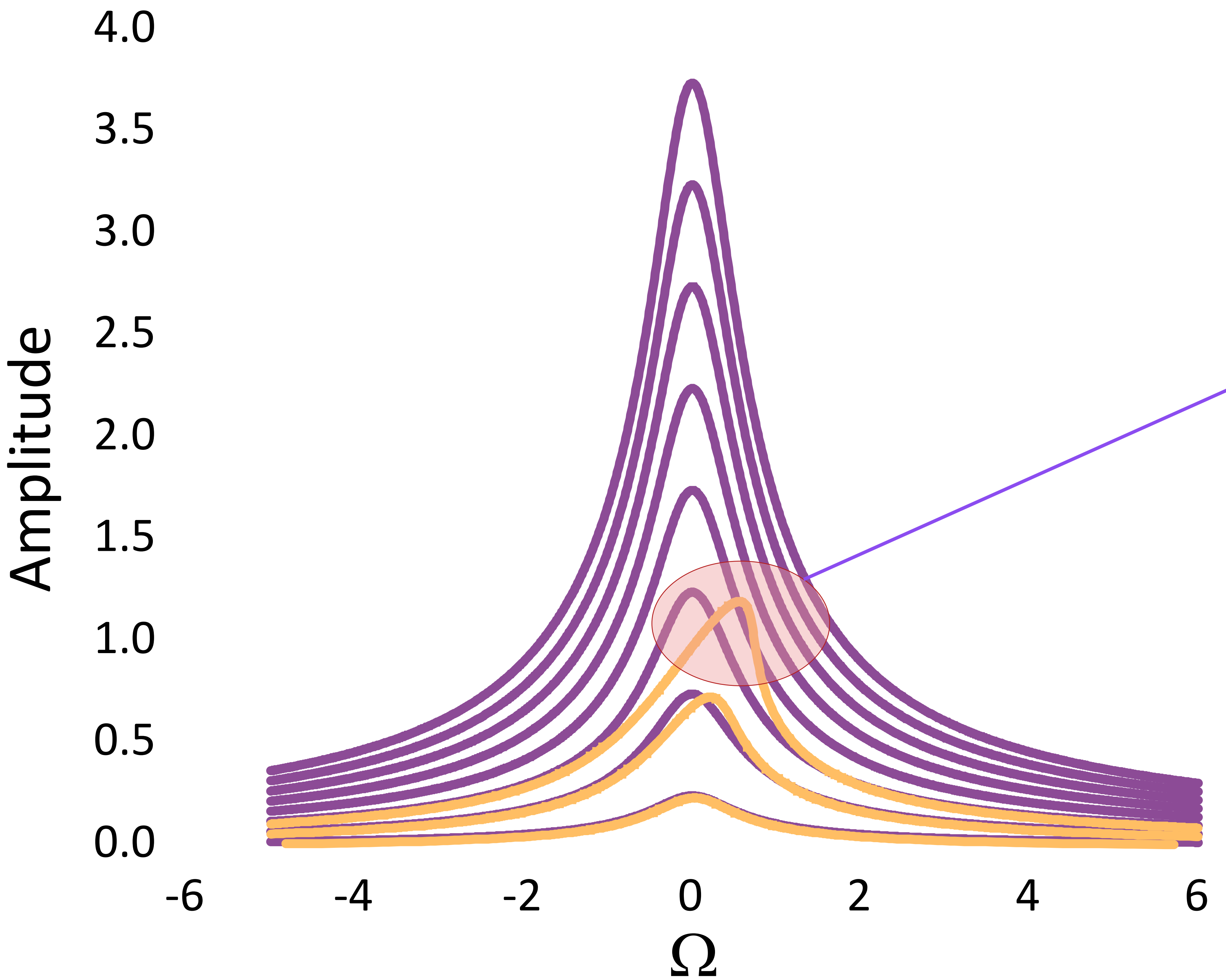
- Linear: $$m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$$
- Nonlinear: $$m\ddot{z} + \Gamma\dot{z} + kz + \alpha z^3 + \eta z^2\dot{z} = G\cos(\omega t)$$

  - Bistability and histeretical behaviour



Critical amplitude – Onset of non linearity

- Linear: $m\ddot{z} + \Gamma\dot{z} + kz = G\cos(\omega t)$
- Nonlinear: $m\ddot{z} + \Gamma\dot{z} + kz + \alpha z^3 + \eta z^2\dot{z} = G\cos(\omega t)$



When detuning of maximum is the linewidth

Onset of nonlinearity

$$a_c \propto \frac{\mathbf{1}}{\sqrt{Q}}$$

Critical amplitude – Onset of non linearity

- Detuning is the linewidth

$$a_c \propto \frac{1}{\sqrt{Q}}$$

- As a function of dimensions
  - Unstressed clamped-clamped beam, Membrane

$$\boldsymbol{a_c \propto thickness}$$

  - Cantilever, highly stressed beam or membranes

$$\boldsymbol{a_c \propto length}$$

  - Capacitive actuation

$$\boldsymbol{a_c \propto gap}$$

  - Acoustic resonators (material NL)

$$\boldsymbol{a_c \propto t, w, L}$$

The smaller the resonator, the earlier it goes nonlinear

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6.4 – Resonators Parametric drive, Composite beams, Surface effects

ME-426 – Micro/Nanomechanical Devices

Prof. Guillermo Villanueva

École polytechnique fédérale de Lausanne

Introduction

- Parametric drive
- Composite beams
- Surface effects

Parametric drive

Parametric Drive

- Alternative actuation technique
  - No force at ω_n is applied
  - ω_n (via k, m) is tuned at **twice the natural frequency ($2\boldsymbol{\omega_n}$)**
  - Motion occurs above an instability threshold

$$m\ddot{z} + \Gamma\dot{z} + \boldsymbol{k}\left(\mathbf{1}+\frac{\boldsymbol{h}}{\boldsymbol{Q}}\cos(\omega_P t)\right)z + \alpha z^3 + \eta z^2\dot{z} = G_D\cos(\omega_D t)$$

Parametric Drive

$$m\ddot{z} + \Gamma\dot{z} + \boldsymbol{k}\left(\boldsymbol{1} + \frac{\boldsymbol{h}}{\boldsymbol{Q}}\cos(\omega_P t)\right)z + \alpha z^3 + \eta z^2\dot{z} = G_D\cos(\omega_D t)$$

$$\frac{dA}{dT} + \frac{\tilde{\gamma}}{2}A - i\frac{\tilde{h}}{4}A^* e^{i\Omega_P T} - \frac{1}{8}(3\tilde{\alpha}i - \eta)|A|^2 A = -i\frac{g}{2}e^{i\Omega_D T}$$

Composite Beams

Frequency of composite beams

$$\omega_n^2 = \frac{EI}{\rho A}\frac{1}{L^4}\frac{\int_0^1 \phi_n''^2(\xi)\,d\xi}{\int_0^1 \phi_n^2(\xi)\,d\xi}\left(1+\frac{\sigma A}{EI}L^2\frac{\int_0^1 \phi_n'^2(\xi)\,d\xi}{\int_0^1 \phi_n''^2(\xi)\,d\xi}\right) = \frac{EI}{\rho A}\frac{1}{L^4}\beta_n^2\left(1+\frac{\sigma A}{EI}L^2\gamma_n\right)$$

$$\langle EI\rangle = \int_A E(y,z)(z-z_0)^2 dydz$$

$$\langle \rho A\rangle = \int_A \rho(y,z) dydz$$

$$\langle \sigma A\rangle = \int_A \sigma(y,z) dydz$$

$$\omega_n^2 = \frac{\langle \boldsymbol{EI}\rangle}{\langle \boldsymbol{\rho A}\rangle}\frac{1}{L^4}\frac{\int_0^1 \phi_n''^2(\xi)\,d\xi}{\int_0^1 \phi_n^2(\xi)\,d\xi}\left(1+\frac{\langle \boldsymbol{\sigma A}\rangle}{\langle \boldsymbol{EI}\rangle}L^2\frac{\int_0^1 \phi_n'^2(\xi)\,d\xi}{\int_0^1 \phi_n''^2(\xi)\,d\xi}\right) = \frac{\langle \boldsymbol{EI}\rangle}{\langle \boldsymbol{\rho A}\rangle}\frac{1}{L^4}\beta_n^2\left(1+\frac{\langle \boldsymbol{\sigma A}\rangle}{\langle \boldsymbol{EI}\rangle}L^2\gamma_n\right)$$

Equation of motion

$$m_{eff,n}\ddot{w}_n(t) + \frac{m_{eff,n}\omega_n}{Q_n}\dot{w}_n(t) + k^{\sigma}_{eff,n}w_n(t) = F_{eff,n}(t)$$

$$m_{eff,n} = \langle\rho A\rangle L \int_0^1 \phi_n^2(\xi)\, d\xi$$

$$Q_n^{-1} = \frac{c}{\langle\rho A\rangle\omega_n}$$

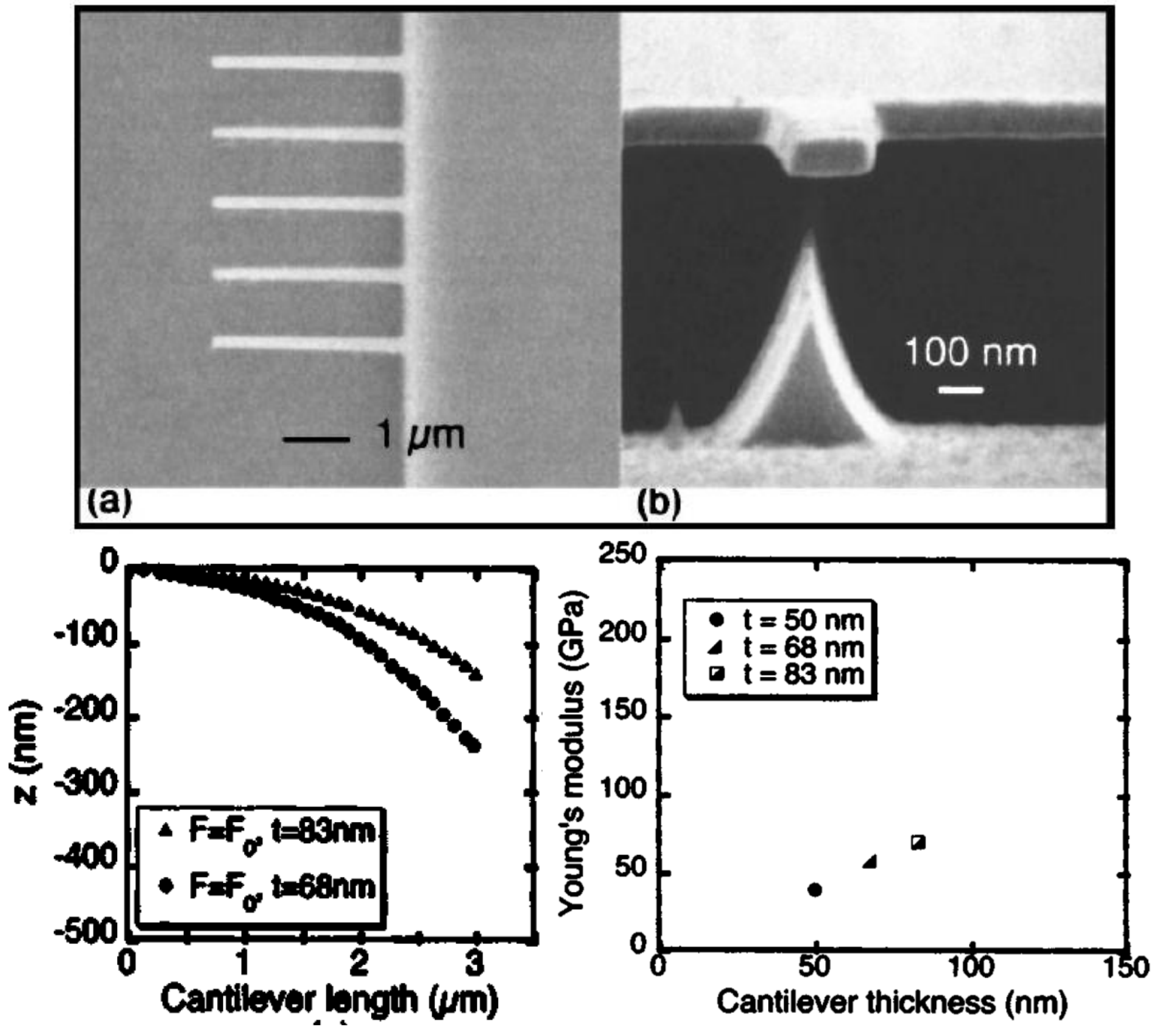
$$k^{\sigma=0}_{eff,n} = \frac{\langle EI\rangle}{L^3}\int_0^1 \phi_n''^2(\xi)\, d\xi$$

$$F_{eff,n} = \int_0^L p(x,t)\phi_n(x)\, dx$$

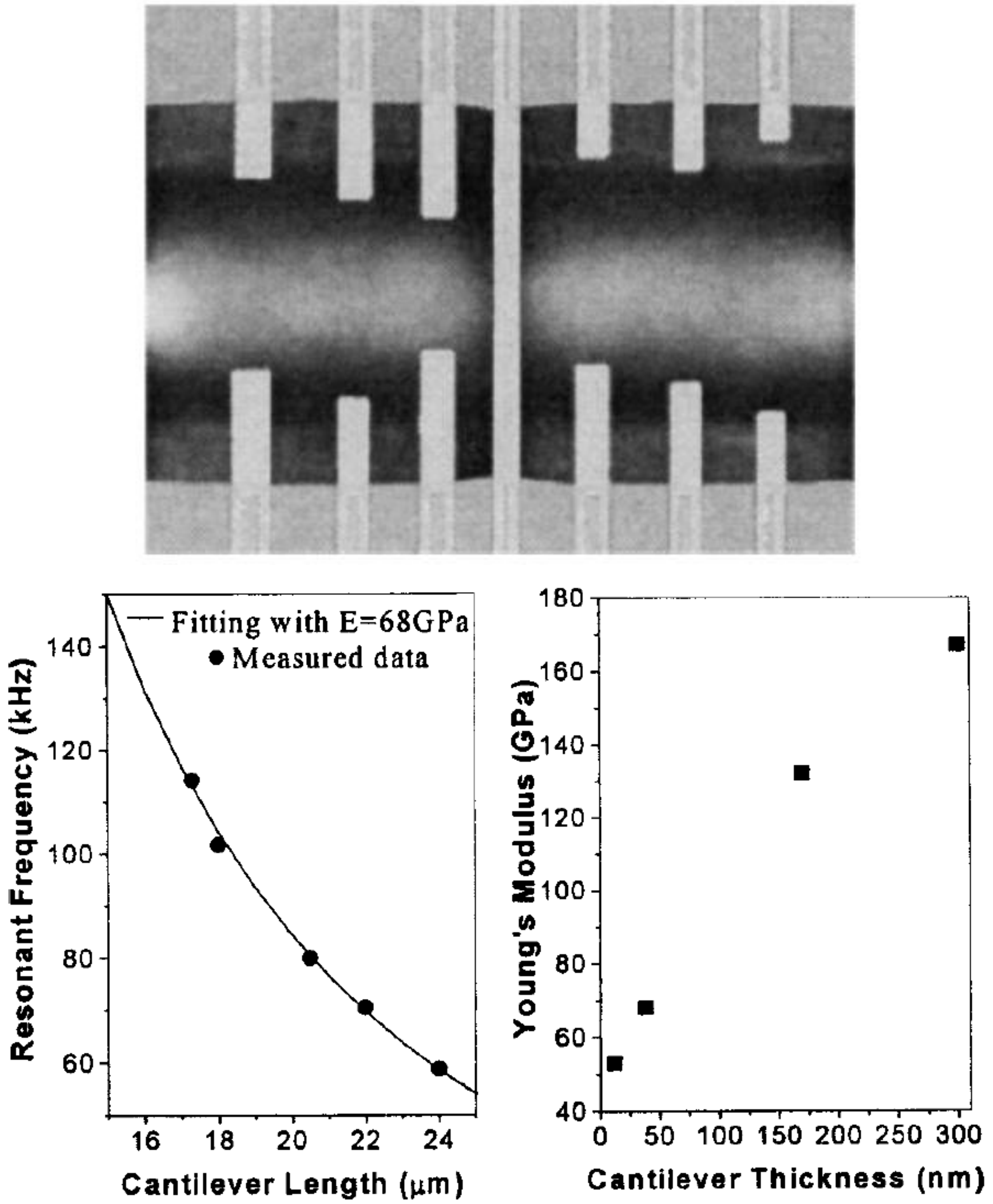
$$\boldsymbol{k^{\sigma}_{eff,n} = \frac{\langle EI\rangle}{L^3}\int_0^1 \phi_n''^2(\xi)\, d\xi + \frac{\langle \sigma A\rangle}{L}\int_0^1 \phi_n'^2(\xi)\, d\xi}$$

Surface Effects

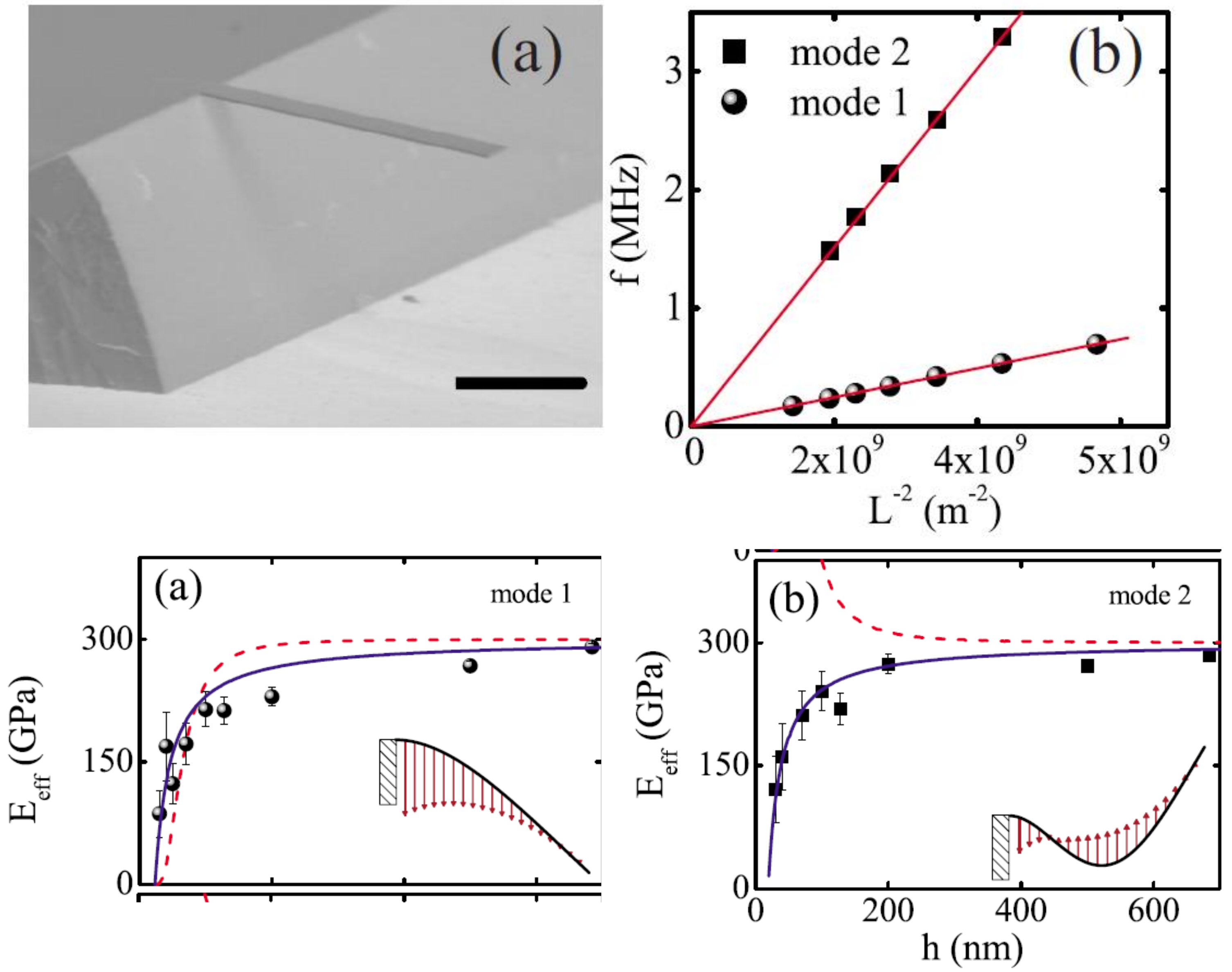
Is Young Modulus the same for thin layers?



- Cr cantilevers
- Force-Deflection experiment
- Apparent drop in Young's mod
- https://doi.org/10.1063/1.1807945



- Si cantilevers
- Resonance frequency
- Apparent drop in Young's mod
- https://doi.org/10.1063/1.1618369



- SiN cantilevers
- Resonance frequency
- Apparent drop in Young's mod
- https://doi.org/10.1063/1.3152772

Surface-Volume ratio is key

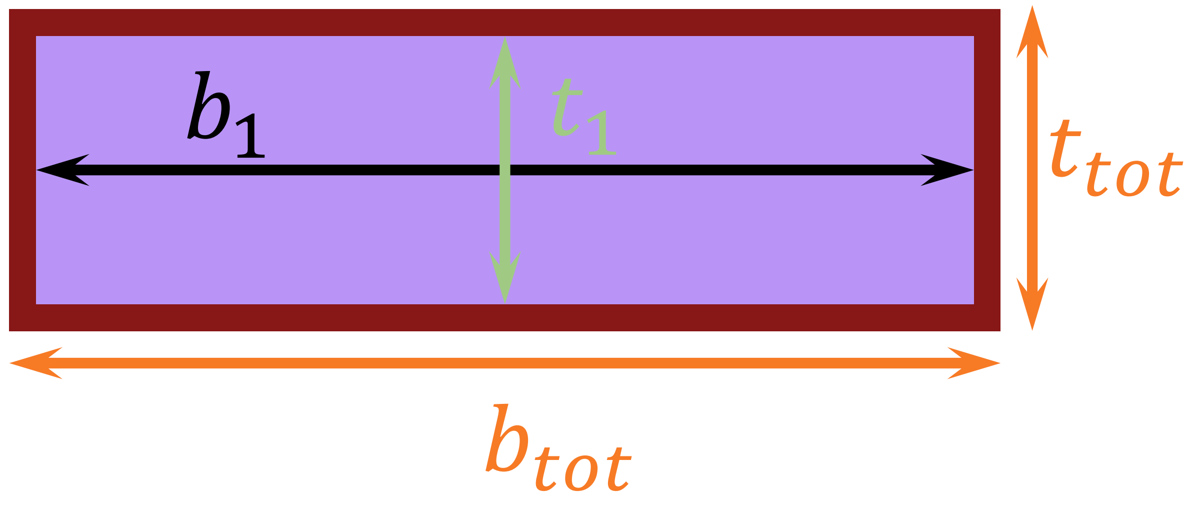
- Surface modification is difficult to avoid (e.g. oxidation of materials in air)
- When thickness reduces, the surface state becomes more and more important

$$f_{1,cant} = \sqrt{\frac{\langle EI \rangle}{\langle \rho A \rangle}} \frac{0.56}{L^2}$$

$$k_{cant} = \frac{3\langle EI \rangle}{L^3}$$

$$\langle EI \rangle = \int_A E(y,z)(z - z_0)^2 dydz$$

$$\langle \rho A \rangle = \int_A \rho(y,z) dydz$$



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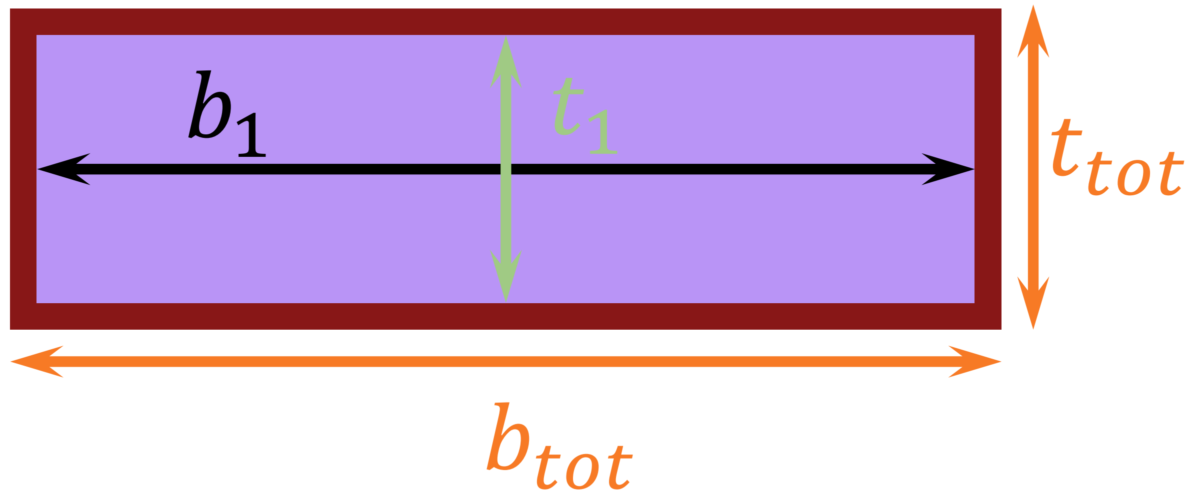

$$\langle EI \rangle = E_1\frac{b_1 t_1^3}{12} + E_2\left(\frac{b_{tot}t_{tot}^3}{12} - \frac{b_1 t_1^3}{12}\right) = E_1\frac{b_{tot}t_{tot}^3}{12} + (E_2 - E_1)\left(\frac{b_{tot}t_{tot}^3}{12} - \frac{b_1 t_1^3}{12}\right) \approx$$

$$\approx E_1\frac{b_{tot}t_{tot}^3}{12} + (E_2 - E_1)\frac{b_1 t_1^3}{12}2\frac{\delta}{t_1} \approx E_1 I_{tot}\left(1 + 2\frac{\delta}{t}(E_2 - E_1)\right)$$

$$\langle \rho A \rangle = \rho_1 b_1 t_1 + \rho_2(b_{tot}t_{tot} - bt_1) \approx \rho_1 A_{tot}\left(1 + 2\frac{\delta}{t}(\rho_2 - \rho_1)\right)$$